Information Theory and Coding
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EPFL Winter Semester 2008/2009
Handout #20, Tuesday, 02 November 2008

Homework Set #9

Due 15 December 2008, before 12:00 noon, INR 031/032/038

Problem 1 (Conditional differential entropy of Gaussian random vectors)

Consider the zero-mean jointly Gaussian random variables X and Y with covariance matrix

$$\mathbb{E}\left[\begin{pmatrix} X \\ Y \end{pmatrix}\begin{pmatrix} X & Y \end{pmatrix}\right] = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix},$$

i.e.,

$$(X,Y) \sim p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho xy}{\sigma_x\sigma_y}\right)\right).$$

(a) Find $f(x|y)$.

(b) Using (a), find $h(X|Y)$.

(c) Interpret $h(X|Y)$ for $\rho = 0$ and $\rho = 1$.

(d) Now assume that $(\mathbf{X}, \mathbf{Y})$ are jointly Gaussian random vectors with zero mean and covariance $\mathbb{E}\left[\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}\begin{pmatrix} \mathbf{X^t Y^t} \end{pmatrix}\right] = \begin{pmatrix} \mathbf{K_{11}} & \mathbf{K_{12}} \\ \mathbf{K_{12}} & \mathbf{K_{22}} \end{pmatrix}$, where $\mathbf{X^t}$ is the transpose of the $\mathbf{X}$. Find $f(\mathbf{X}|\mathbf{Y})$.

(e) Use (d) to find $h(\mathbf{X}|\mathbf{Y})$.

Problem 2 (Parallel Gaussian Channel)

Consider the Gaussian channel shown in Fig. 2 where $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_2 \sim \mathcal{N}(0, N_2)$ are independent Gaussian random variables and $Y_i = X_i + Z_i$.

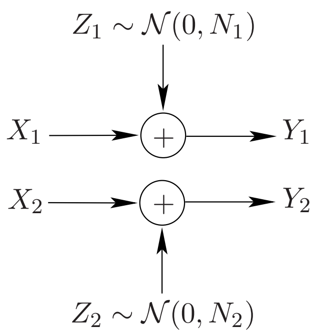

Figure 1: Parallel Gaussian channels

We wish to allocate power to the two parallel channels. Let β_1 and β_2 be fixed. Consider a total cost constraint $\beta_1 P_1 + \beta_2 P_2 \leq \beta$, where P_i is the power allocated to the i-th channel and β_i is the cost per unit power in the channel. Thus, $P_1 \geq 0$ and $P_2 \geq 0$ can be chosen subject to the cost constraint β.

(a) For what value of β does the channel stop acting like a single channel and start acting like a pair of channels?

(b) Evaluate the capacity and find P_1 and P_2 that achieve capacity for $\beta_1 = 1$, $\beta_2 = 2$, $N_1 = 3$, $N_2 = 2$ and $\beta = 10$.

Problem 3 (Two-look Gaussian Channel)

Consider the ordinary Gaussian channel with two correlated looks at X, that is, $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(0, \mathbf{K})$, where

$$\mathbf{K} = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

Find the capacity for

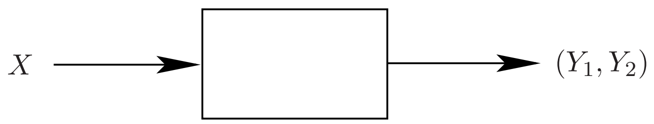

Figure 2: Two-look Gaussian channel

(a) $\rho = 1$

(b) $\rho = 0$

(c) $\rho = -1$

Problem 4 (Intermittent additive noise channel)

Consider the channel $Y_i = X_i + Z_i$, where X_i is the transmitted signal with average power constraint P, Z_i is independent additive noise, and Y_i is the received signal. Let

$$Z_i = \begin{cases} 0 & \text{with probability } \frac{1}{10} \\ Z^* & \text{with probability } \frac{9}{10}, \end{cases}$$

where $Z^* \sim \mathcal{N}(0, N)$. Thus, Z has a mixture of a Gaussian distribution and a degenerate distribution with mass 1 at 0.

(a) What is the capacity of this channel? This should be a pleasant surprise.

(b) How would you signal to achieve capacity?