Problem 1 (Conditional Differential Entropy of Gaussian Random Vectors)

Consider the zero-mean jointly Gaussian random variables $X$ and $Y$ with covariance matrix

$$E \left[ \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} \right] = \begin{pmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{pmatrix},$$

i.e.,

$$(X,Y) \sim p(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp \left(-\frac{1}{2(1-\rho^2)} \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2\rho \frac{xy}{\sigma_x \sigma_y} \right) \right).$$

(a) Find $f(x|y)$.

(b) Using (a), find $h(X|Y)$.

(c) Interpret $h(X|Y)$ for $\rho = 0$ and $\rho = 1$.

(d) Now assume that $(X,Y)$ are jointly Gaussian random vectors with zero mean and covariance

$$E \left[ \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X^t & Y^t \end{pmatrix} \right] = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix},$$

where $X^t$ is the transpose of the $X$. Find $f(X|Y)$.

(e) Use (d) to find $h(X|Y)$.

Problem 2 (Parallel Gaussian Channel)

Consider the Gaussian channel shown in Fig. 2 where $Z_1 \sim \mathcal{N}(0,N_1)$ and $Z_2 \sim \mathcal{N}(0,N_2)$ are independent Gaussian random variables and $Y_i = X_i + Z_i$.

Figure 1: Parallel Gaussian channels
We wish to allocate power to the two parallel channels. Let $\beta_1$ and $\beta_2$ be fixed. Consider a total cost constraint $\beta_1 P_1 + \beta_2 P_2 \leq \beta$, where $P_i$ is the power allocated to the $i$-th channel and $\beta_i$ is the cost per unit power in the channel. Thus, $P_1 \geq 0$ and $P_2 \geq 0$ can be chosen subject to the cost constraint $\beta$.

(a) For what value of $\beta$ does the channel stop acting like a single channel and start acting like a pair of channels?

(b) Evaluate the capacity and find $P_1$ and $P_2$ that achieve capacity for $\beta_1 = 1$, $\beta_2 = 2$, $N_1 = 3$, $N_2 = 2$ and $\beta = 10$.

**Problem 3 (Two-Look Gaussian Channel)**

Consider the ordinary Gaussian channel with two correlated looks at $X$, that is, $Y = (Y_1, Y_2)$, where

\[
\begin{align*}
Y_1 &= X + Z_1 \\
Y_2 &= X + Z_2
\end{align*}
\]

with a power constraint $P$ on $X$, and $(Z_1, Z_2) \sim \mathcal{N}_2(0, K)$, where

\[
K = \begin{bmatrix}
N & N \rho \\
N \rho & N
\end{bmatrix}.
\]

Find the capacity for

\[
\begin{array}{c}
X \\
\downarrow \\
(Y_1, Y_2)
\end{array}
\]

Figure 2: Two-look Gaussian channel

(a) $\rho = 1$

(b) $\rho = 0$

(c) $\rho = -1$

**Problem 4 (Intermittent Additive Noise Channel)**

Consider the channel $Y_i = X_i + Z_i$, where $X_i$ is the transmitted signal with average power constraint $P$, $Z_i$ is independent additive noise, and $Y_i$ is the received signal. Let

\[
Z_i = \begin{cases}
0 & \text{with probability } \frac{1}{10} \\
Z^* & \text{with probability } \frac{9}{10},
\end{cases}
\]

where $Z^* \sim \mathcal{N}(0, N)$. Thus, $Z$ has a mixture of a Gaussian distribution and a degenerate distribution with mass 1 at 0.

(a) What is the capacity of this channel? This should be a pleasant surprise.

(b) How would you signal to achieve capacity?