FINAL EXAM

Saturday January 19, 2008, 14:15-18:00
This exam has 5 problems and 100 points in total.

Instructions

- You are allowed to use 2 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

GOOD LUCK!

Problem 1

[**25 pts**]

Let $x[n]$ and $y[n]$ be two sequences that have finite support. More precisely, $x[n] = 0$ for $n < 0$ and $n > N-1$, and $y[n] = 0$ for $n < 0$ and $n > M-1$. Assume that $M > N$. Also, assume that none of the two sequences is the all-zero sequence.
Now, let $v[n]$ be the (non-circular) discrete convolution of $x[n]$ with $y[n]$:

$$v[n] = (x * y)[n].$$

[*8pts*] (a) Given the information above, what is the maximum length of the support of $v[n]$?

[*2pts*] (b) Compute the discrete-time Fourier transform (DTFT) of $v[n]$. Give the result in terms of $x[n]$ and $y[n]$.

[*15pts*] (c) Assume a teaching assistant claims that he has found $x[n]$ and $y[n]$, with $N = 7$ and $M = 10$, such that M uniformly spaced samples of the DTFT of $v[n]$ are all zero. More precisely, he claims that

$$\tilde{V}[k] = \sum_{n=-\infty}^{\infty} v[n] e^{-j\frac{2\pi}{M}kn} = 0$$

for all $k \in \{0, \ldots, M-1\}$.

Given the information above (about $x[n]$, $y[n]$ and $v[n]$), is it possible that the teaching assistant tells the truth or can you be sure that he is lying? If he can be right, give an example of $x[n]$ and $y[n]$ (with the given values for N and M) for which the claim is true. If he is lying, give a proof.

Problem 2

[**17 pts**]

Consider the system given in Figure 1, where

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $h[n]$ the impulse response of the overall system, *i.e.*, $y[n] = x[n] * h[n]$.

[*5pts*] (a) Write the difference equation of the system, *i.e.*, express $y[n]$ as a function of a finite number of terms in $x[n], x[n-1], \ldots$ and $y[n-1], y[n-2], \ldots$.

[*2pts*] (b) Compute the system function $H(z)$, which is the Z-transform of the impulse response $h[n]$.

[*2pts*] (c) Determine all the zeros and poles of $H(z)$. For each zero and each pole, indicate its multiplicity.

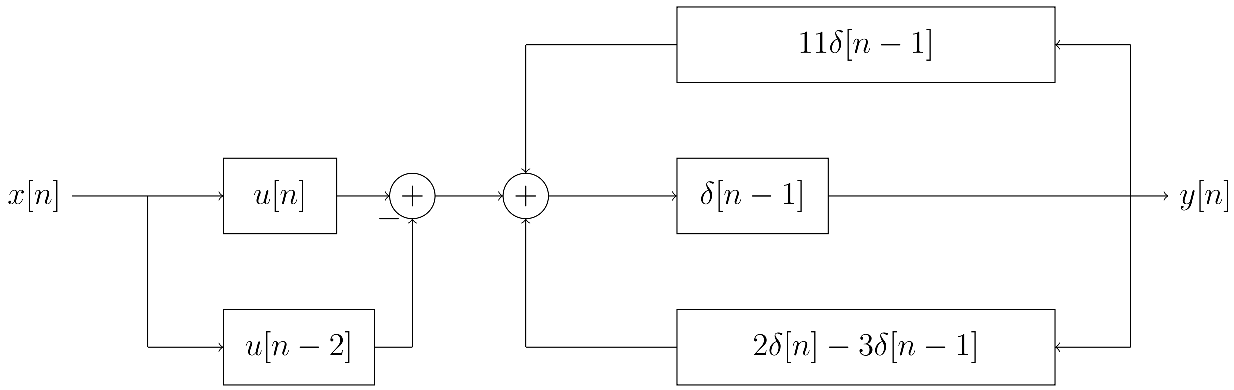

Figure 1: System with Impulse Response $h[n]$.

[*5pts*] (d) Compute $h[n]$. (Assume that $y[n] = 0$ for $n < 0$.)
Hint: In case you find a Z-transform of the form $H(z) = z^{-1}\tilde{H}(z)$, you can first determine $\tilde{h}[n]$.

[*1pts*] (e) Is the system linear?

[*1pts*] (f) Is the system time-invariant?

[*1pts*] (g) Is the system stable?

Problem 3

[25 pts]

Consider a continuous-time signal $x_c(t)$, and assume that its continuous-time Fourier transform (CTFT) is as given in Figure 2.

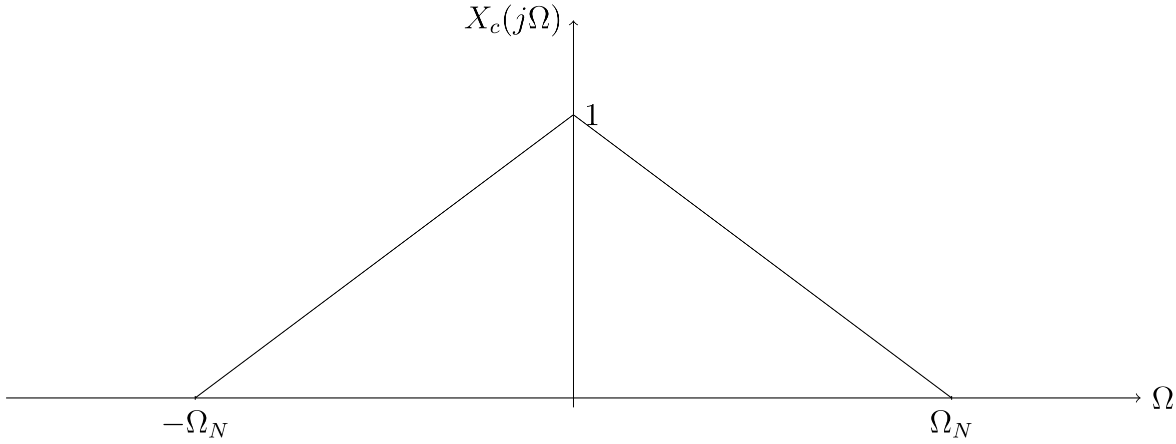

Figure 2: Spectrum of $x_c(t)$.

[*1pts*] (a) According to the sampling theorem, what is the smallest sampling frequency Ω_s at which reconstruction of $x_c(t)$ from the samples is possible?

[*1pts*] (b) Assume that we sample $x_c(t)$ at a sampling frequency $\Omega'_s = 3\Omega_N$. Let $u[n]$ be the sample sequence, *i.e.*, $u[n] = x_c(nT'_s)$, where $T'_s = \frac{2\pi}{\Omega'_s}$. Is perfect reconstruction of $x_c(t)$ possible from the sample sequence $u[n]$?

[*4pts*] (c) Sketch the discrete-time Fourier transform (DTFT) $U(e^{j\omega})$ of the sample sequence $u[n]$ for $\omega \in [-3\pi, 3\pi]$. Make sure to label all the important points on both axes.

[*3pts*] (d) Assume that we upsample $u[n]$ by a factor of 2 to obtain $v[n]$, *i.e*,

$$v[n] = \begin{cases} u[\frac{n}{2}] & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Sketch the DTFT $V(e^{j\omega})$ of the new sequence $v[n]$ for $\omega \in [-3\pi, 3\pi]$. Again, label all the important points on both axes.

[*6pts*] (e) Now, we use sinc interpolation on $v[n]$ to construct a continuous-time signal $y_c(t)$ (using $\Omega'_s = 3\Omega_N$). In other words,

$$y_c(t) = \sum_{n=-\infty}^{\infty} v[n] \operatorname{sinc}(\frac{t - nT'_s}{T'_s}).$$

Sketch the CTFT $Y_c(j\Omega)$ of $y_c(t)$. Make sure to label all the important points on both axes.

[*1pts*] (f) According to the sampling theorem, what is the smallest sampling frequency $\tilde{\Omega}_s$ at which reconstruction of $y_c(t)$ from the samples is possible?

[*3pts*] (g) Assume that we sample $y_c(t)$ at a sampling frequency $\hat{\Omega}_s = 2\Omega_N$ to obtain the sample sequence $w[n]$. Sketch the DTFT $W(e^{j\omega})$ of the sample sequence $w[n]$ for $\omega \in [-3\pi, 3\pi]$. Make sure to label all the important points on both axes.

[*6pts*] (h) Is perfect reconstruction of $y_c(t)$ possible from the sample sequence $w[n]$? If yes, explain how. If not, explain why. (You can give your answer in terms of diagrams and some explanations.)

Problem 4

[**17 pts**]

Consider the system given in Figure 3, where $\mathcal{S}$ is a time-**variant** operator which acts on its input $u[n]$ as follows:

$$v[n] = \mathcal{S}\{u[n]\} = \begin{cases} u[n] & \text{if } n \text{ is even} \\ \frac{1}{2}u[n] + \frac{1}{2}u[n-1] & \text{if } n \text{ is odd.} \end{cases}$$

[*7pts*] (a) Compute $v[n]$ in terms of $x[n]$ and $y[n]$.
Hint: First write down $v[0]$, $v[1]$, $v[2]$ and $v[3]$, and then generalize.

[*5pts*] (b) Rewrite the system under the form given in Figure 4. All you are allowed to do is fill in the gaps in the rectangles and circles. All the filters that you use should be linear and time-invariant.
Hint: You are allowed to put $\delta[n]$ in some of the filters.

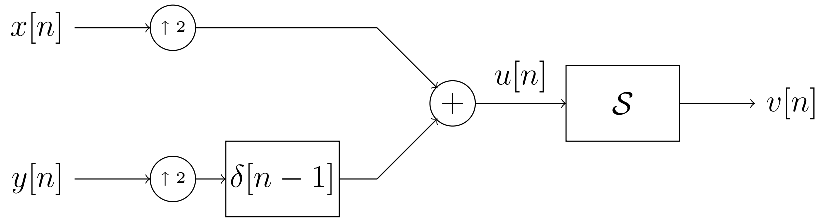

Figure 3: Multirate system

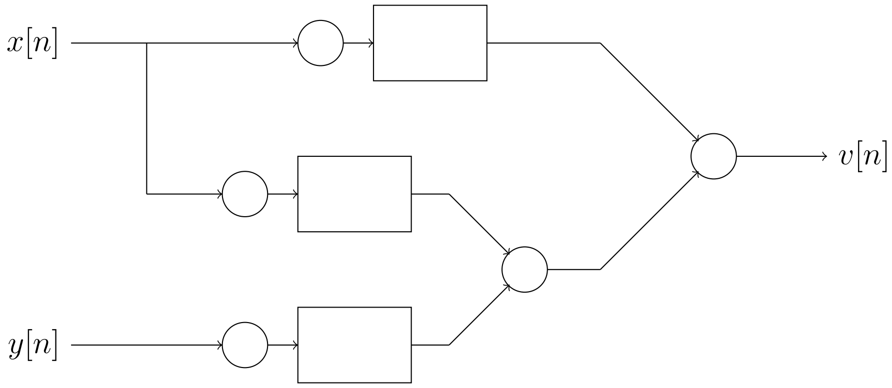

Figure 4: Equivalent multirate system

[*1pts*] (c) Compute the Z-transform $V(z)$ of the system output as a function of $X(z)$ and $Y(z)$.

[*2pts*] (d) Now we feed $v[n]$ into the system given in Figure 5. Compute $a[n]$ and $b[n]$. Express the result in terms of $x[n]$ and $y[n]$.

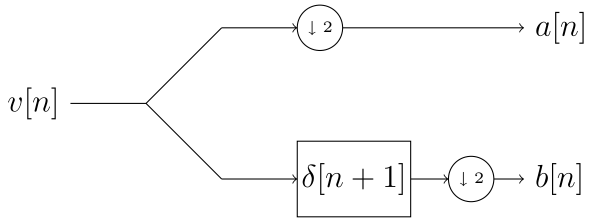

Figure 5: Continuation of the multirate system

[*2pts*] (e) Find a system that inverts the system in Figure 3, *i.e.*, your system should have input $v[n]$ and outputs $x[n]$ and $y[n]$. Draw the system diagram and determine all the operations and filters used. All the filters that you use should be linear and time-invariant.

Problem 5

[**16 pts**]

Consider the filter bank in Figure 6, where the filter functions are the Haar filters:

$$h_0[n] = \frac{1}{\sqrt{2}} \begin{cases} 1 & \text{if } n = 0, -1 \\ 0 & \text{otherwise.} \end{cases}$$

$$h_1[n] = \frac{1}{\sqrt{2}} \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

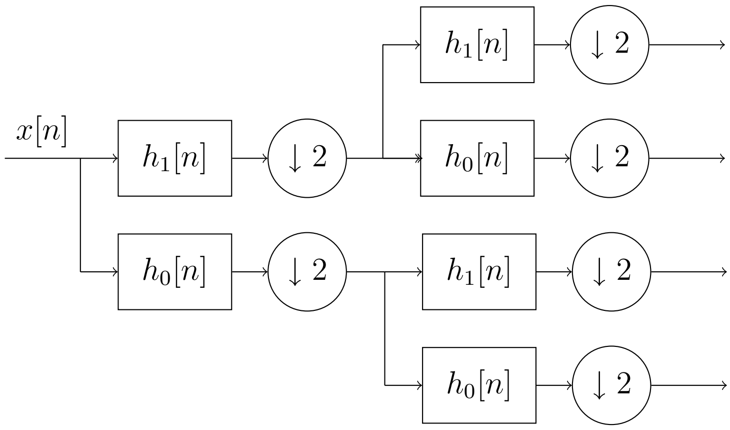

Figure 6: Tree structured filterbank

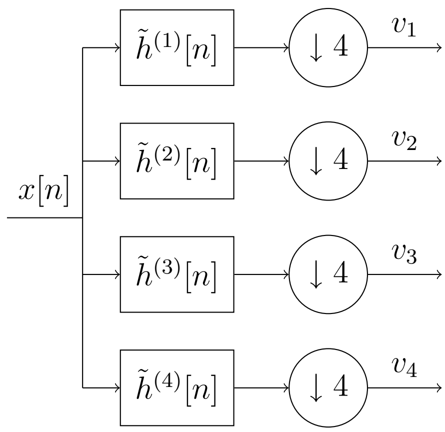

Figure 7: Equivalent filter bank with constant downsampling rate

[*6pts*] (a) Write the equivalent filters $\tilde{h}^{(1)}, \ldots, \tilde{h}^{(4)}$ such that the four outputs of the system in Figure 7 are the same as in Figure 6.

[*3pts*] (b) Assume that $x[n]$ is of length 4 and that it is fed into the filterbank in Figure 6. Let $\mathbf{v} = [v_1, v_2, v_3, v_4]^T$ be a vector containing the outputs of the four branches. Also, let $\mathbf{x} = [x[0], x[1], x[2], x[3]]^T$ be a vector representation of the input sequence. Find a matrix $\mathbf{M}$ such that

$$\mathbf{v} = \mathbf{Mx}.$$

[*7pts*] (c) Now, we construct a new filterbank, using Haar filters, which is such that

$$\mathbf{v} = \mathbf{M}'\mathbf{x},$$

where the transfer matrix is

$$\mathbf{M}' = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Draw the system which implements this new filterbank.