

Solutions: Homework Set # 8

**Problem 1** (ORDER OF UP- AND DOWNSAMPLING)

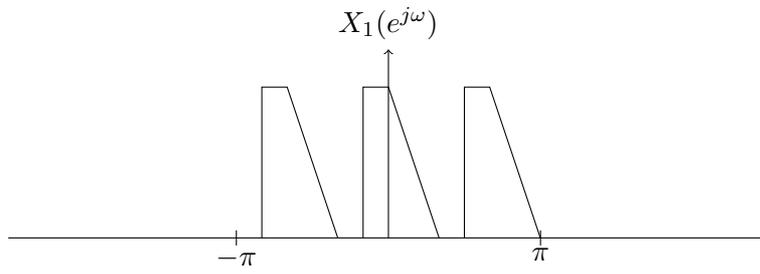
- (a) The magnitudes of  $X_1(e^{j\omega})$  and  $Y_1(e^{j\omega})$  are given on Figures 1a and 1b, respectively.
- (b) The magnitudes of  $X_2(e^{j\omega})$  and  $Y_2(e^{j\omega})$  are given on Figures 1c and 1d, respectively.
- (c) From Figures 1b and 1d, we can see that  $Y_1(e^{j\omega}) = Y_2(e^{j\omega})$ .
- (d) For the first system, we have

$$\begin{aligned} X_1(e^{j\omega}) &= X(e^{j3\omega}) \\ Y_1(e^{j\omega}) &= \frac{1}{2} \left[ X_1(e^{j\omega/2}) + X_1(e^{j(\omega/2+\pi)}) \right] \\ &= \frac{1}{2} \left[ X(e^{j3\omega/2}) + X(e^{j(3\omega/2+\pi)}) \right]. \end{aligned}$$

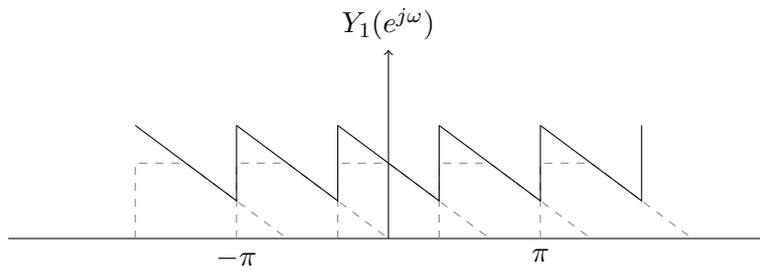
For the second system, we have

$$\begin{aligned} X_2(e^{j\omega}) &= \frac{1}{2} \left[ X(e^{j\omega/2}) + X(e^{j(\omega/2+\pi)}) \right] \\ Y_2(e^{j\omega}) &= X_2(e^{j3\omega}) \\ &= \frac{1}{2} \left[ X(e^{j3\omega/2}) + X(e^{j(3\omega/2+\pi)}) \right]. \end{aligned}$$

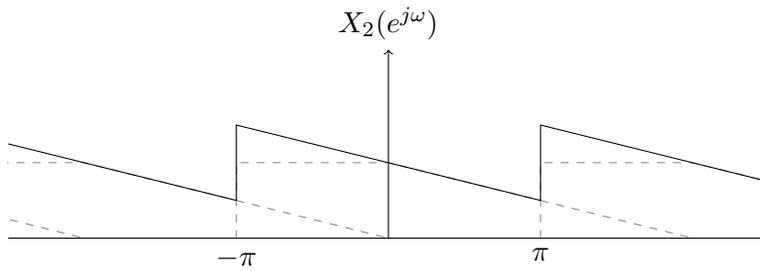
- (e) From the results of (e), we see that for general  $x[n]$ ,  $Y_1(e^{j\omega}) = Y_2(e^{j\omega})$ .
- (f) 1. Suppose  $L$  and  $M$  have a common factor  $Q$ , *i.e.*,  $L = L'Q$  and  $M = M'Q$ . Then for  $k = M'$  we have  $kL = M'L'Q = L'M$ , and so  $W_M^{kL} = W_M^0 = 1$ .  
 But we already have  $W_M^k = 1$  for  $k = 0$ , so a 1 occurs twice in the sequence  $\{W_M^{kL}\}_{k=0}^{M-1}$ , and therefore this sequence cannot be a permutation. It follows that  $Y_1(z) \neq Y_2(z)$ .
2. Suppose that  $L$  and  $M$  are coprime. By Bezout's identity there exist  $m$  and  $n$  such that  $mL + nM = 1$ , *i.e.*,  $mL \bmod M = 1$ . Then,  $mlL \bmod M = l$  for  $l \in \{0, \dots, M-1\}$ . Thus, for any  $l$ , if we set  $k = ml \bmod M$  then  $kL \bmod M = l$ .
3. From the above it follows that  $\{W_M^{kL}\}_{k=0}^{M-1}$  and  $\{W_M^k\}_{k=0}^{M-1}$  are permutations, and so  $Y_1(z) = Y_2(z)$ , if and only if  $L$  and  $M$  are coprime.



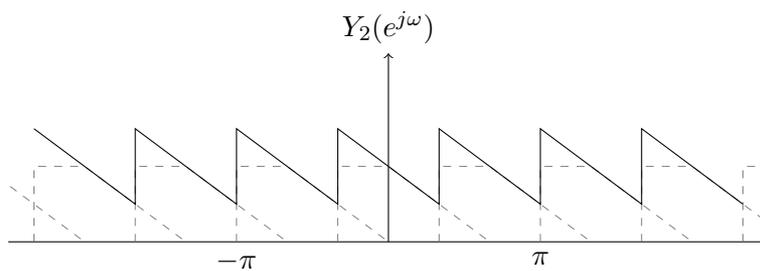
(a) The DTFT  $X_1(e^{j\omega})$  for Problem 1(a).



(b) The DTFT  $Y_1(e^{j\omega})$  for Problem 1(a).



(c) The DTFT  $X_2(e^{j\omega})$  for Problem 1(b).



(d) The DTFT  $Y_2(e^{j\omega})$  for Problem 1(b).

Figure 1: Illustrations for Problems 1(a) and 1(b).

## Problem 2

- (a) Using the properties of the  $z$ -transform of up-sampling and down-sampling, we can easily write

$$\begin{aligned} U(z) &= X(z^L) \\ V(z) &= z^{-N}U(z) = z^{-N}X(z^L) \\ Y(z) &= \frac{1}{M} \sum_{k=0}^{M-1} V(e^{\frac{2\pi j}{M}k} z^{\frac{1}{M}}) = \frac{1}{M} \sum_{k=0}^{M-1} e^{-\frac{2\pi j}{M}kN} z^{-\frac{N}{M}} X(e^{\frac{2\pi j}{M}kL} z^{\frac{L}{M}}) \end{aligned} \quad (1)$$

- (b) By assuming  $M = M'd$  and  $L = L'd$ , and replacing them in (1), we have

$$\begin{aligned} Y(z) &= \frac{1}{M'd} \sum_{k=0}^{M'd-1} e^{-\frac{2\pi j}{M'd}kN} z^{-\frac{N}{M'd}} X(e^{\frac{2\pi j}{M'd}kL'd} z^{\frac{L'd}{M'd}}) \\ &= \frac{1}{M'd} z^{-\frac{N}{M'd}} \sum_{q=0}^{M'-1} \sum_{p=0}^{d-1} e^{-\frac{2\pi j}{M'd}(pM'+q)N} X(e^{\frac{2\pi j}{M'}(pM'+q)L'} z^{\frac{L'}{M'}}) \\ &= \frac{1}{M'd} z^{-\frac{N}{M'd}} \sum_{q=0}^{M'-1} \sum_{p=0}^{d-1} e^{-\frac{2\pi j}{d}pN} e^{-\frac{2\pi j}{M'd}qN} X(e^{\frac{2\pi j}{M'}qL'} z^{\frac{L'}{M'}}) \\ &= \frac{1}{M'd} z^{-\frac{N}{M'd}} \sum_{q=0}^{M'-1} \left[ e^{-\frac{2\pi j}{M'd}qN} X(e^{\frac{2\pi j}{M'}qL'} z^{\frac{L'}{M'}}) \sum_{p=0}^{d-1} e^{-\frac{2\pi j}{d}pN} \right]. \end{aligned} \quad (2)$$

Now if  $N = N'd$ , we have  $e^{-\frac{2\pi j}{d}pN} = e^{-2\pi jN'p} = 1$ , and

$$\begin{aligned} Y(z) &= \frac{1}{M'd} z^{-\frac{N'd}{M'd}} \sum_{q=0}^{M'-1} \left[ e^{-\frac{2\pi j}{M'd}qN'd} X(e^{\frac{2\pi j}{M'}qL'} z^{\frac{L'}{M'}}) d \right] \\ &= \frac{1}{M'} \sum_{q=0}^{M'-1} e^{-\frac{2\pi j}{M'}qN'} z^{-\frac{N'}{M'}} X(e^{\frac{2\pi j}{M'}qL'} z^{\frac{L'}{M'}}) \end{aligned}$$

Clearly the  $z$ -transform of the system shown in Fig. 5 is similar to (1), where  $M$ ,  $L$ ,  $N$  are replaced by  $M'd$ ,  $L'd$ , and  $N'd$ , respectively. The resulting  $z$ -transform is exactly equal to (3). Therefore, the systems are equivalent.

- (c) If  $d \nmid N$ , the inner summation in (2) can be written as

$$\sum_{p=0}^{d-1} e^{-\frac{2\pi j}{d}pN} = \frac{e^{-\frac{2\pi j}{d}dN} - 1}{e^{-\frac{2\pi j}{d}N} - 1} = 0.$$

Thus  $Y(z) = 0$ .

## Problem 3

- (a) We call the signals produced in the system as shown in Fig. 2. Using these notations, and properties of the spectrum of the downsampled signal, we have the spectrum shown on Figure 3, where  $X'(e^{j\omega})$  is in fact the input of the second layered of the tree.

Similarly, we can draw the spectrum of the other signals as shown in Figure 4.

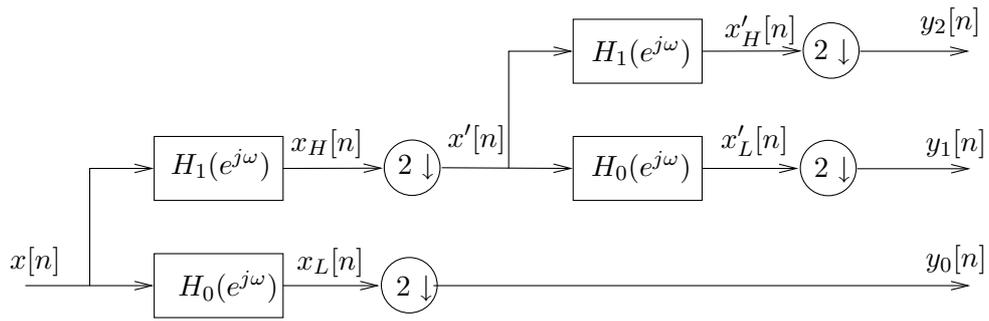


Figure 2: Tree-structured Analysis filter bank

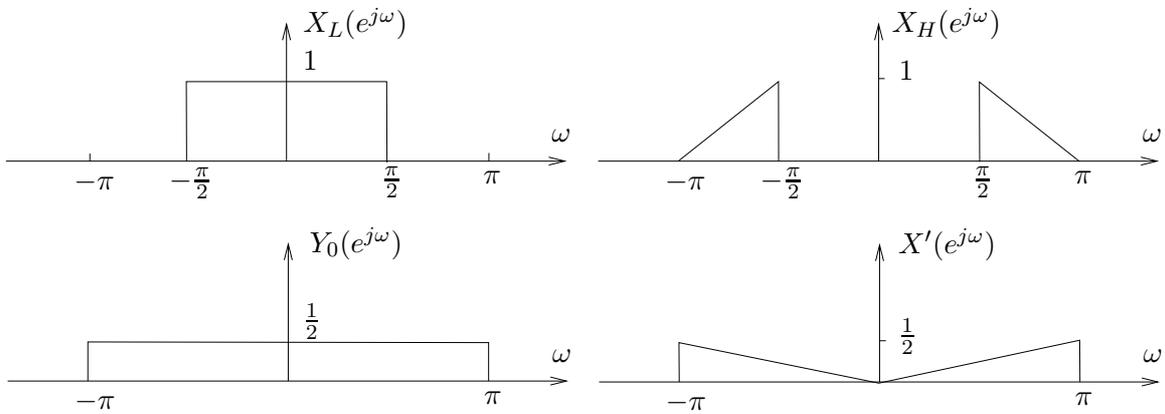


Figure 3: Spectrums

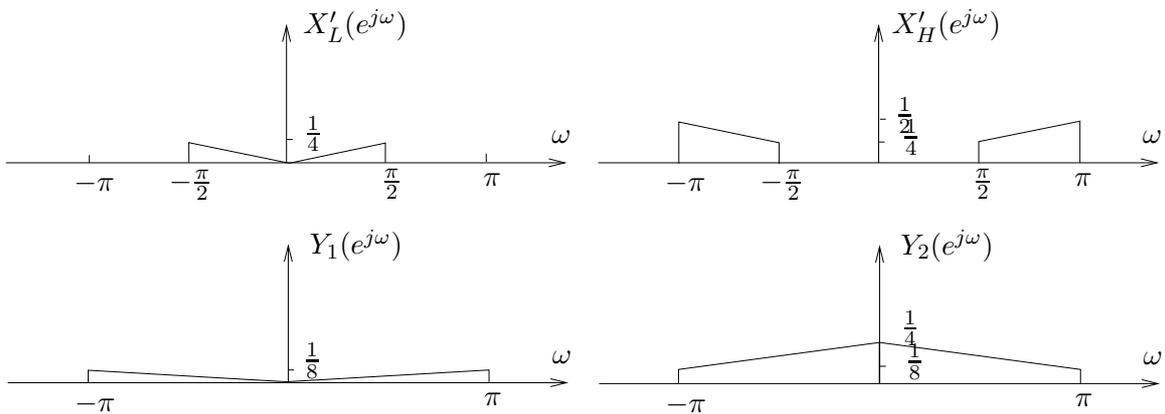


Figure 4: Spectrums

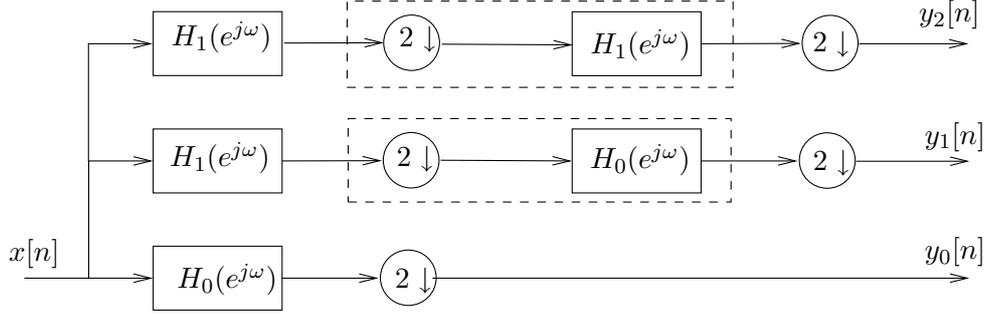


Figure 5: Equivalent system to tree-structured analysis filter bank (1)

(b) Clearly the system shown in Fig. 6 is equivalent to the one in Fig. 5.

Using the Nobel identity, the part of the system in dashed lines can be converted as shown in Figure 6, where

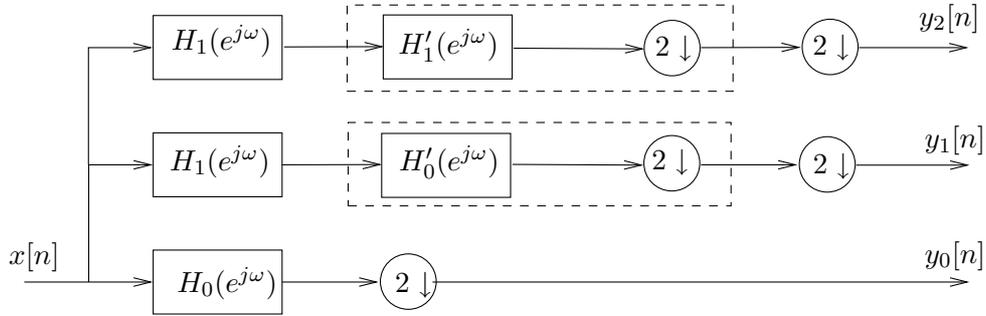


Figure 6: Equivalent system to tree-structured analysis filter bank (2)

$$H'_0(e^{j\omega}) = H_0(e^{2j\omega}) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \leq \frac{3\pi}{4} \\ 1 & \frac{3\pi}{4} < |\omega| \leq \pi, \end{cases} \quad (3)$$

and

$$H'_1(e^{j\omega}) = H_1(e^{2j\omega}) = \begin{cases} 0 & |\omega| \leq \frac{\pi}{4} \\ 1 & \frac{\pi}{4} < |\omega| \leq \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < |\omega| \leq \pi. \end{cases} \quad (4)$$

Finally the system can be rewritten as in Figure 7, where  $G_0(e^{j\omega}) = H_0(e^{j\omega})$ ,

$$G_1(e^{j\omega}) = H_1(e^{j\omega})H'_0(e^{j\omega}) = \begin{cases} 0 & |\omega| \leq \frac{\pi}{4} \\ 1 & \frac{3\pi}{4} < |\omega| \leq \pi, \end{cases}$$

and

$$G_2(e^{j\omega}) = H_1(e^{j\omega})H'_1(e^{j\omega}) = \begin{cases} 0 & |\omega| \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < |\omega| \leq \frac{3\pi}{4}, \\ 0 & \frac{3\pi}{4} < |\omega| \leq \pi. \end{cases}$$

Finally, you can easily see that  $M_0 = 2$ ,  $M_1 = 4$ , and  $M_2 = 4$ .

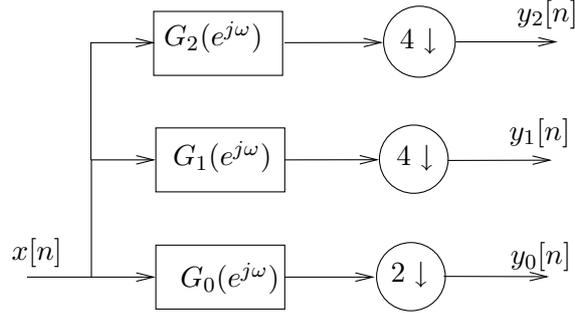


Figure 7: Equivalent system to tree-structured analysis filter bank (3)

- (c) Recall the sufficient conditions for exact reconstruction, you have in the course. It is easy to see that the analysis filters in this problem are orthogonal. However, they are not normalized and have not unit energy. Therefore, we can get the reconstruction filters, by just scaling the original constructing filters, to compensate the loss of energy. Let  $F_i(e^{j\omega}) = 2H_i(e^{-j\omega})$ , for  $i = 0, 1$ . Since  $H_i(e^{j\omega})$ 's are symmetric with respect to the vertical axis, these filters can be explicitly rewritten as

$$F_0(e^{j\omega}) = \begin{cases} 2 & |\omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \leq \pi, \end{cases} \quad F_1(e^{j\omega}) = \begin{cases} 0 & |\omega| \leq \frac{\pi}{2} \\ 2 & \frac{\pi}{2} < |\omega| \leq \pi. \end{cases}$$

You can easily check that the output of the synthesis filter would be exactly the same as  $x[n]$ .

#### Problem 4 (BLOCK DFT)

- (a) First note that by definition of  $x^{(i)}[n]$ , we have

$$x[n] = \sum_{i=0}^{Q-1} x^{(i)}[n].$$

Let  $I_i = \{iN, iN + 1, \dots, (i + 1)N - 1\}$ . The sequence  $x^{(i)}[n]$  is nonzero only for  $n \in I_i$ . Therefore, if we write the reconstruction of block  $N$  as

$$x^{(i)}[n] = \sum_{k=0}^{N-1} X^{(i)}[k] \varphi_k^{(i)}[n], \quad (5)$$

we see that  $\varphi_k^{(i)}[n]$  must be equal to zero for  $n \notin I_i$ . For  $n \in I_i$ ,  $\varphi_k^{(i)}[n]$  must be the  $k^{\text{th}}$  DFT basis vector. Hence,

$$\varphi_k^{(i)}[n] = \begin{cases} \frac{1}{N} e^{j \frac{2\pi k n}{N}}, & n \in I_i \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

- (b) Consider  $\langle \varphi_k^{(i)}, \varphi_l^{(j)} \rangle = \sum_n \left( \varphi_k^{(i)}[n] \right)^* \varphi_l^{(j)}[n]$ . If  $i \neq j$ , the two basis functions have disjoint support sets and hence their scalar product is zero. If  $i = j$  and  $k \neq l$ , orthogonality follows from the orthogonality of the DFT basis.

(c) For  $N = 2$ , the basis functions are

$$\varphi_0^{(i)}[n] = \frac{1}{2} \begin{cases} 1 & n = 2i, 2i + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_1^{(i)}[n] = \frac{1}{2} \begin{cases} 1 & n = 2i \\ -1 & n = 2i + 1 \\ 0 & \text{otherwise} \end{cases}$$

These are equal to the Haar basis functions, up to the different scaling factor; for the Haar basis functions the scaling factor is  $1/\sqrt{2}$ . This in fact makes the Haar basis an *orthonormal basis*.

(d) We want to find  $h_k[n]$  such that  $v_k[n] = X^{(n)}[k]$ . The signal on branch  $k$  after downsampling is

$$\begin{aligned} v_k[n] &= x_k[Nn] \\ &= \sum_l h_k[l]x[Nn - l] \\ &= \sum_l h_k[-l]x[Nn + l], \end{aligned} \tag{7}$$

where in the last equality we have substituted  $l$  by  $-l$ . Note that the sum goes over all  $l \in \mathbb{Z}$ , as we don't know a-priori the length of the filter.

On the other hand, the  $k^{\text{th}}$  coefficient of the DFT of block  $n$  is

$$X^{(n)}[k] = \sum_{l=0}^{N-1} x[nN + l]e^{-j\frac{2\pi}{N}kl}.$$

Comparing the right hand side with the right hand side of (7), we see that for  $v_k[n] = X^{(n)}[k]$ , we need

$$h_k[l] = \begin{cases} e^{j\frac{2\pi}{N}kl} & l = -(N-1), \dots, -1, 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is important to note that here the basis is orthogonal but *not orthonormal*, therefore we don't have  $h_k[n] = \varphi_k^{(0)}[-n]$  as we would have in the orthonormal case.

Next, we want to determine  $g_k[n]$  such that  $\hat{x}[n] = x[n]$ . This is true if and only if the two sequences are equal in every block, i.e., for all  $i$ ,

$$\hat{x}[iN + l] = x^{(i)}[iN + l], \quad i = 0, \dots, N-1. \tag{8}$$

Writing explicitly the expression for  $\hat{x}[iN + l]$  as the sum of the filter outputs, we get

$$\begin{aligned} \hat{x}[iN + l] &= \sum_{k=0}^{N-1} (g_k * y_k)[iN + l] \\ &= \sum_{k=0}^{N-1} \sum_j g_k[j]y_k[iN + l - j]. \end{aligned} \tag{9}$$

Again, the convolution sum is over all  $j \in \mathbb{Z}$ . Because  $y_k[n]$  is the result of upsampling  $v_k[n]$  by  $N$  we have (using the formula for upsampling seen in class)

$$\begin{aligned} y_k[n] &= \sum_m v_k[m] \delta[n - mN] \\ &= \sum_m X^{(m)}[k] \delta[n - mN]. \end{aligned}$$

Replacing this for  $y_k[n]$  in (9), we obtain

$$\hat{x}[iN + l] = \sum_{k=0}^{N-1} \sum_j g_k[j] \sum_m X^{(m)}[k] \delta[(i - m)N + l - j].$$

For  $j \neq (i - m)N + l$  this is zero, so we can remove the sum over  $j$  to get

$$\hat{x}[iN + l] = \sum_{k=0}^{N-1} \sum_m g_k[(i - m)N + l] X^{(m)}[k].$$

Now, if we compare this to the block reconstruction formula (5), we see that if we set  $g_k[n] = \varphi_k^{(0)}[n]$ , then we have

$$\begin{aligned} \hat{x}[iN + l] &= \sum_{k=0}^{N-1} \sum_m \varphi_k^{(0)}[(i - m)N + l] X^{(m)}[k] \\ &= \sum_{k=0}^{N-1} \varphi_k^{(0)}[l] X^{(i)}[k] \\ &= x^{(i)}[iN + l], \end{aligned}$$

where the second equality follows from the finite support of  $\varphi_k^{(0)}[n]$  (cf. (6)). Therefore, the condition for perfect reconstruction (8) is satisfied for  $g_k[n] = \varphi_k^{(0)}[n]$ .

## Problem 5

(Sampling Rate Change and Aliasing)

(a) In Matlab, we type:

```
>> [vista,fs] = wavread('vista.wav');
>> soundsc(vista,fs)
>> fs
```

The sampling frequency is  $f_s = 22050$ [Hz].

(b) Note that  $\frac{13230}{22050} = \frac{3}{5}$ . Hence, we can first upsample by a factor 3 and then downsample by a factor 5. Note that to minimize aliasing effects, it is better to first do the upsampling and then the downsampling.

```
>> vista_u3 = upsample(vista,3);
>> vista_u3d5 = downsample(vista_u3,5);
>> plot(linspace(-pi,pi,length(vista_u3d5)),abs(fftshift(fft(vista_u3d5))))
```

The spectrum of “vista\_u3d5” has 3 copies of the original spectrum. See Figure 9. As a comparison, we plot the spectrum of the original sequence in Figure 8.

```
>> plot(linspace(-pi,pi,length(vista)),abs(fftshift(fft(vista))))
```

The 2 high-frequency copies of the spectrum of “vista\_u3d5” are due to aliasing. If one listens to the audio-sequence, one can hear the high-frequency sounds corresponding to the 2 additional copies (like a wistling background noise).

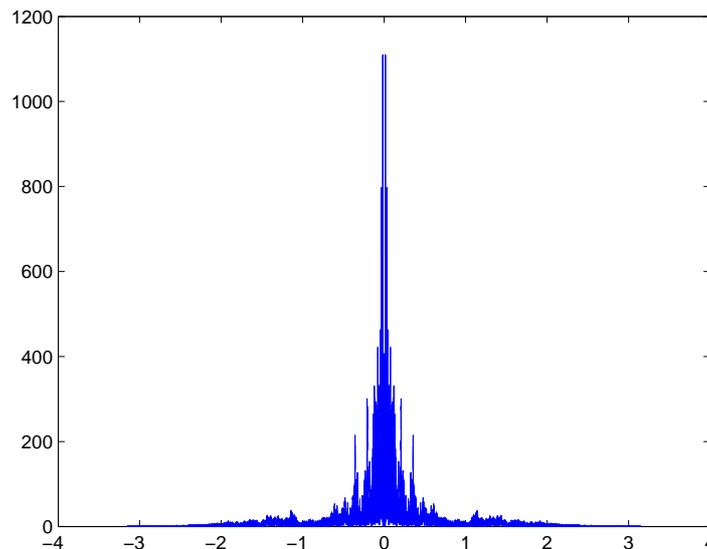


Figure 8: Magnitude of the DFT of the original sequence at a sampling rate of 22050.

- (c) To combat this severe aliasing effect, we use antialiasing filters. After upsampling, we apply a low-pass filter with a cut-off at  $\frac{\pi}{3}$ :

```
>> [hpm1,err] = firpm(40,[0 0.31 0.35 1], [1 1 0 0], [1 10]);
>> vista_u3f = conv(vista_u3,hpm1);
```

Then, before doing the downsampling, we apply another low-pass filter with a cut-off at  $\frac{\pi}{5}$ :

```
>> [hpm2,err] = firpm(40,[0 0.18 0.22 1], [1 1 0 0], [1 10]);
>> vista_u3ff = conv(vista_u3f,hpm2);
>> vista_u3ffd5 = downsample(vista_u3ff,5);
>> plot(linspace(-pi,pi,length(vista_u3ffd5)),abs(fftshift(fft(vista_u3ffd5))))
```

The plot of “vista\_u3ffd5” in Figure 10 is now quite similar to the original spectrum, except that it is wider (exactly by a factor of  $\frac{5}{3}$  wider).

```
>> soundsc(vista_u3ffd5,fs*3/5)
```

Also the sound is much nicer now. It is not too different from the original.

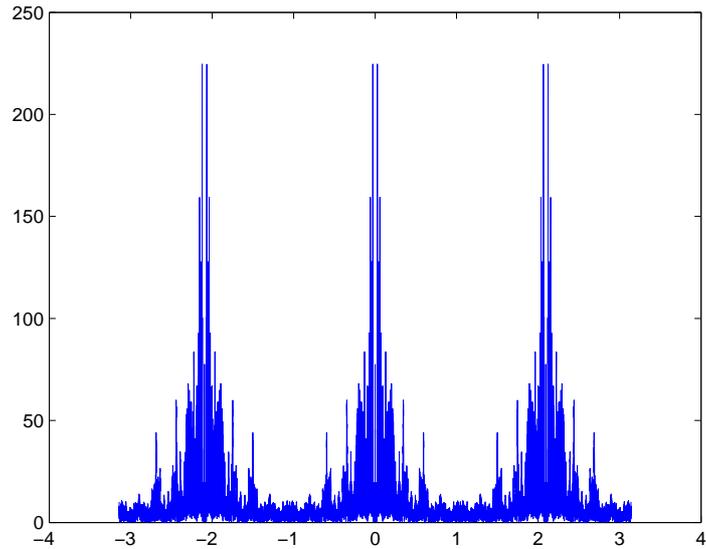


Figure 9: Magnitude of the DFT of the sequence at a sampling rate of 13230 (no antialiasing).

- (d) In Figures 11 to 15, we explain what is going on. In any case, the sequence “vista” is first upsampled by a factor 3 (Figure 12). If we do not immediately apply a low-pass filter with cutoff  $\frac{\pi}{3}$ , there will be very serious aliasing problems. After applying such a filter, we obtain the spectrum shown in Figure 13, corresponding to the sequence “vista\_u3f”. Now, assume that we do not apply the low-pass filter with cutoff  $\frac{\pi}{5}$ , but that we directly downsample the sequence “vista\_u3f”. In this case, we obtain the spectrum in Figure 14, which again suffers from aliasing. If, however, we apply the antialiasing filter before downsampling, then we cut the spectrum in Figure 13 to have only support  $[-\frac{\pi}{5}, \frac{\pi}{5}]$ . After downsampling, we then obtain the sequence “vista\_u3fd5”, which corresponds, in the interval  $[-\pi, \pi]$ , to a stretched version of the original spectrum (stretching-factor  $\frac{5}{3}$ ). We can see that both aliasing effects increase the power in the high frequencies ( $\omega$  close to  $-\pi$  or  $\pi$ ). We can hear these frequencies as noise in “vista\_u3d5”. If the antialiasing is done correctly, there is no increase of high frequencies, and hence no noise. However, very high frequencies (above  $0.6\pi$  in the original sequence) are cut off. But it is hard to detect this by listening to the audio signal.
- (e) By reading the help output, we can see that `interp` combines upsampling with the correct (perfect) antialiasing filter. `decimate` does the same for downsampling.

```
>> help interp
>> help decimate
```

Hence, applying the two functions should have the same effect as what we did in part (c).

```
>> soundsc(decimate(interp(vista,3),5),fs*3/5)
```

The result sounds the same as the result in (c).

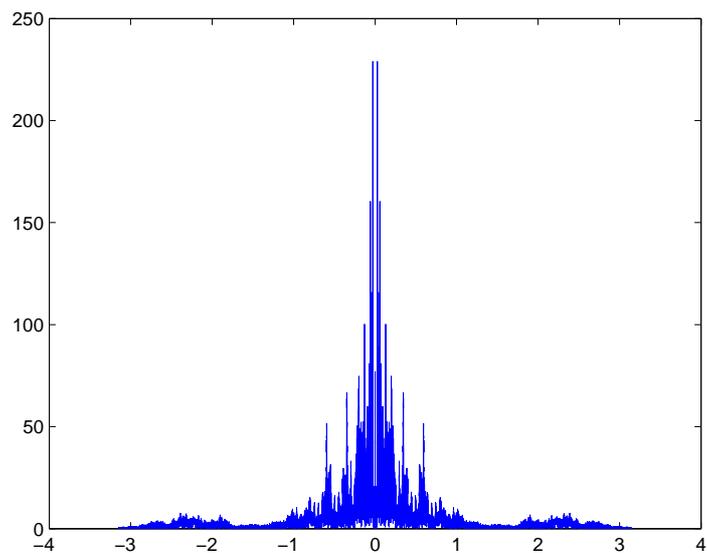


Figure 10: Magnitude of the DFT of the sequence at a sampling rate of 13230 (with antialiasing filters).

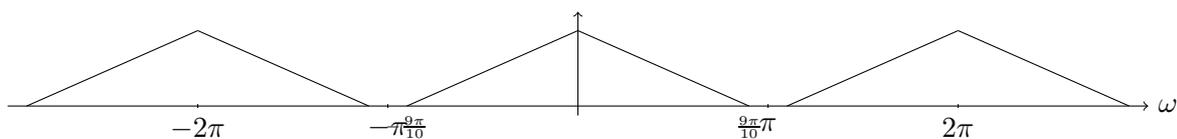


Figure 11: Spectrum of the original sequence "vista".

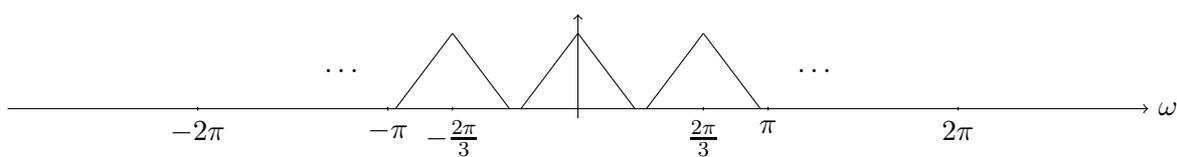


Figure 12: Spectrum of the upsampled sequence "vista\_u3".

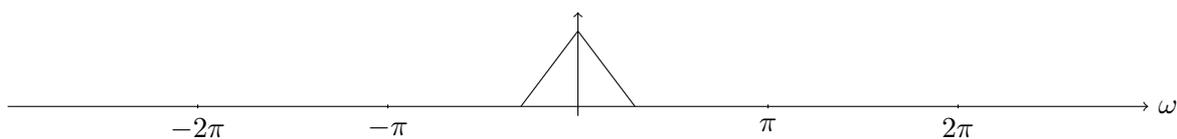


Figure 13: Spectrum of the upsampled sequence after the antialiasing filter ("vista\_u3f").

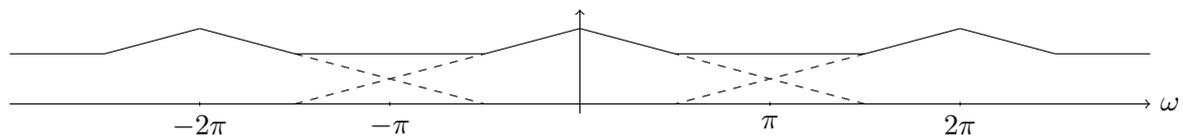


Figure 14: Spectrum of the sequence after upsampling (with antialiasing), followed by downsampling without antialiasing.

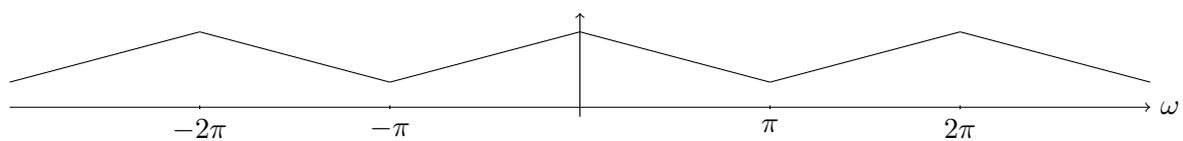


Figure 15: Spectrum of the sequence after upsampling (with antialiasing), followed by downsampling with antialiasing (“vista\_u3ffd5”).