EPFL Winter Semester 2007/2008
Handout # 33, Friday, December 21, 2007

Solutions: Homework Set # 7

Problem 1

(a) We have

$$\begin{aligned} X_c(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{j\Omega t}dt \\ &= \begin{cases} 1 & \Omega \in [-\pi, \pi] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Let $\Omega_s = 2\pi f_s = \pi$. Hence, $\tilde{X}_c(j\Omega)$ is a sum of copies of $X_c(j\Omega)$, each shifted by a multiple of π. It is easy to see that this sum gives a constant, *i.e.*, for any Ω, $\tilde{X}_c(j\Omega) = 2$. (It would be 1 if the Ω_s was 2π.)

(c) We can use the formula from the lecture notes:

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{T_s}\tilde{X}_c(j\frac{\omega}{T_s}) \\ &= \frac{1}{T_s}2 \\ &= \frac{1}{2}2 = 1. \end{aligned}$$

(d) **Note:** Here, we use a sinc-interpolation procedure that corresponds to the sampling frequency f_s. Hence, in the formula of the lecture notes (Section 10.6.3), we set $\Omega_N = \frac{1}{2}\Omega_s$. We find the following formula useful:

$$\begin{aligned} \hat{X}(j\Omega) &= \begin{cases} \frac{\pi}{\Omega_N}X(e^{j\pi\frac{\Omega}{\Omega_N}}) & \text{for } |\Omega| \leq \Omega_N \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{2\pi}{\Omega_s}X(e^{j\pi\frac{2\Omega}{\Omega_s}}) & \text{for } |\Omega| \leq \frac{\Omega_s}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{2\pi}{2\pi\frac{1}{2}}1 & \text{for } |\Omega| \leq \frac{2\pi\frac{1}{2}}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2 & \text{for } |\Omega| \leq \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(e) Note that $\hat{X}(j\Omega) = 2X(j2\Omega)$. Thus, we can compute

$$\begin{aligned}
\hat{x}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2X(j2\Omega)e^{j\Omega t}d\Omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2X(j\theta)e^{j\frac{\theta}{2}t}\frac{1}{2}d\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)e^{j\frac{t}{2}\theta}d\theta \\
&= x(\frac{t}{2}),
\end{aligned}$$

where we have used the variable change $\theta = 2\Omega$. Therefore, we have

$$\hat{x}(t) = \text{sinc}(\frac{t}{2}).$$

This is different from $x(t)$. The difference is clearly due to the aliasing that happens when we sample at a frequency which is too small to allow for perfect reconstruction.

Problem 2

(a) It is clear that $X(j\Omega) = 0$ for $\Omega > 8\pi$, and $\Omega < -12\pi$. So, $\Omega_N = \max\{|8\pi|, |-12\pi|\} = 12\pi$.

(b) After sampling, the magnitude of the spectrum is scaled by $\frac{1}{T_s} = 12$, and it will be repeated by period $\frac{2\pi}{T_s} = 24\pi$. Therefore, the $X(j\Omega)$ would be as shown in Fig. 1.

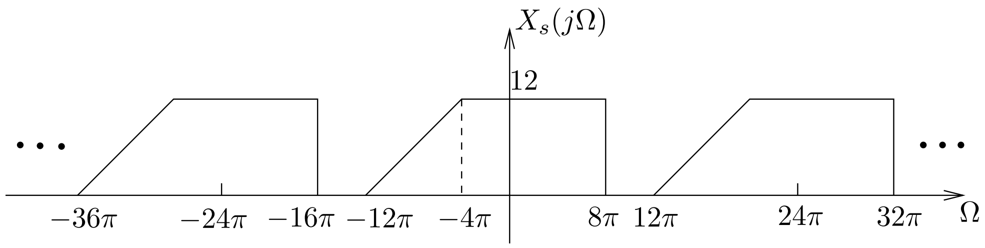


Figure 1: Spectrum of the sampled signal with $T_s = \frac{1}{12}$: $X_s(j\Omega)$

(c) The spectrum of the DTFT of a signal is quit similar to its sampled version, unless the horizontal axis which should be scaled by $T = \frac{1}{12}$. It is shown in Fig. 2.

(d) Since we sample the signal with $\Omega_s = 2\Omega$, we are in the critical point of the Nyquist theorem. For this example since the $X(j\Omega)\mid_{\Omega=12}= 0$, we do not have any aliasing effect, and therefore we can exactly reconstruct the signal. You can check that using an ideal reconstruction filter,

$$H_r(j\Omega) = \begin{cases} 1 & |\Omega| \leq \Omega_c \\ 0 & |\Omega| > \Omega_c, \end{cases}$$

the signal can be reconstructed from its sampled version. Notice that the only possible value for the cut-off frequency of the filter $\omega_c = 12\text{Hz}$, in this example.

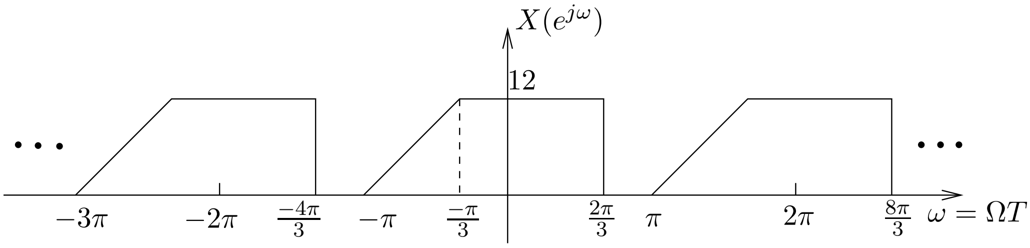


Figure 2: Spectrum of DTFT of the signal, $X(e^{j\omega})$

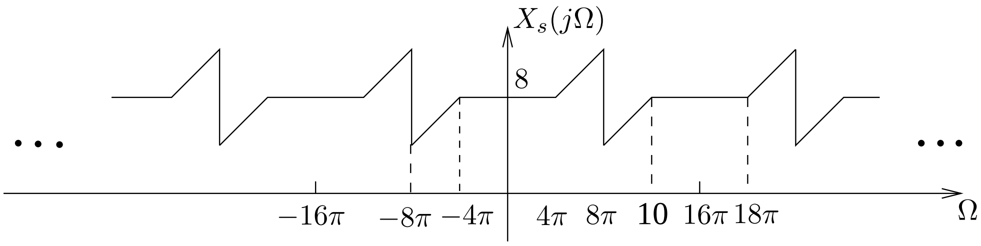


Figure 3: Spectrum of the sampled signal with $T_s = \frac{1}{8}$: $X_s(j\Omega)$

(e) By sampling with sampling time $T_s = \frac{1}{8}$, we get the spectrum shown in Fig. 3. As it is clear form Fig. 3, we have aliasing effect here, and exact reconstruction is impossible.

(f) Using the properties of continuous time Fourier transform, we have

$$Y(j\Omega) = X(j(\Omega - 2))$$

which is in fact the shifted version of $X(j\Omega)$. Fig. 4 shows its spectrum.

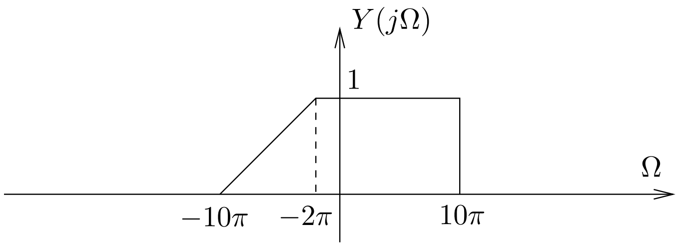


Figure 4: $Y(j\omega)$

(g) Fig. 5 shows the spectrum of the sampled signal, with $T_s = \frac{1}{10}$.

(h) It is clear from the figure that the one-sided bandwidth of $Y(j\Omega)$ is 10π, and since we sample it at $\Omega_s = 20\pi$, we do not have aliasing effect here. The correct way to reconstruct $X(j\Omega)$ (and therefore $x_c(t)$) from $Y_s(j\Omega)$ is to first filter it by

$$H'_r(j\Omega) = \begin{cases} 1 & |\Omega| \leq 10\pi \\ 0 & |\Omega| > 10\pi, \end{cases}$$

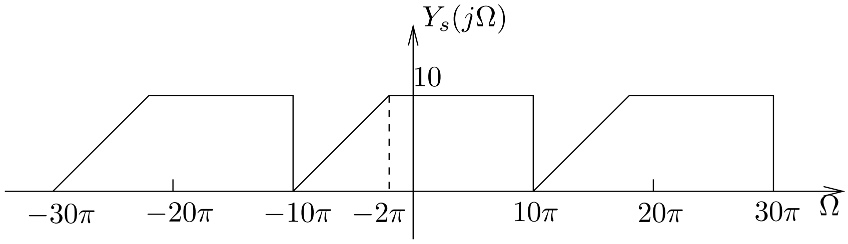


Figure 5: $Y_s(j\omega)$

and shift it back to the right frequency band. That is

$$\begin{aligned} X(j\Omega) &= [Y_s(j\Omega)H_r'(j\Omega)] * \delta(\Omega+2) = Y_s(j(\Omega+2))H_r'(j(\Omega+2)), \\ x_c(t) &= (y_s(t) * h_r'(t))e^{-j2\pi t}. \end{aligned}$$

(i) We saw in part (a) that $\Omega_N = 12\pi$, and therefore, according to Sampling (Nyquist) theorem, we if sample the signal at sampling frequency $\Omega_s > 2\Omega_N$, then we can exactly reconstruct it. This is only a sufficient condition. However, the theorem does not say anything about the necessary condition, and there is no contradiction between the result of part (h) and the theorem.
Remark: You can think of the bandwith as the half of the total length of the frequency interval occupied by the spectrum (for which the spectrum is not zero). By this modified definition, we are also able to do the exact reconstruction. However, this reconstruction might be not that trivial as the original sampling theorem, and it has to be done carefully, *e.g.* the shift pre-filter we used in definition of $Y(j\Omega)$, part (e) of Problem 3, etc.

Problem 3

(a) The condition of the sampling theorem is

$$\Omega_s \geq 2\Omega_N.$$

In our case, $\Omega_N = \Omega_2 = 2\pi f_2$. Hence, the condition is

$$\begin{aligned} 2\pi\frac{1}{T_s} &\geq 22\pi f_2 \\ T_s &\leq \frac{1}{2f_2}. \end{aligned}$$

(b) It is easy to check that

$$T_s' = \frac{1}{f_s'} = \frac{1}{100}[s] > \frac{1}{2f_s} = \frac{1}{400}[s].$$

Hence, f_s' does not satisfy the condition given in part (a).

(c) By the "spectrum" of $v[n]$, we mean the DTFT. To obtain the DTFT of the sampled version of $x_c(t)$, we first sketch

$$X_s(j\Omega) = \frac{1}{T_s'}\sum_{k=-\infty}^{\infty} X_c\big(j(\Omega - k\Omega_s')\big),$$

which is asked for in the next part of this problem. $X_s(j\Omega)$ is just a sum of shifted and scaled copies of $X_c(j\Omega)$ (shifted by multiples of $\Omega_s' = 2\pi \cdot 100$, scaled by $\frac{1}{T_s'}$). It is given in Figure 6. The DTFT of $v[n]$ is now simply

$$V(e^{j\omega}) = X_s(j\frac{\omega}{T_s'}),$$

which basically means that the point $\Omega = \frac{\Omega_s'}{2} = \frac{2\pi}{2T_s'}$ will correspond to $\omega = \pi$. The result is shown in Figure 7.

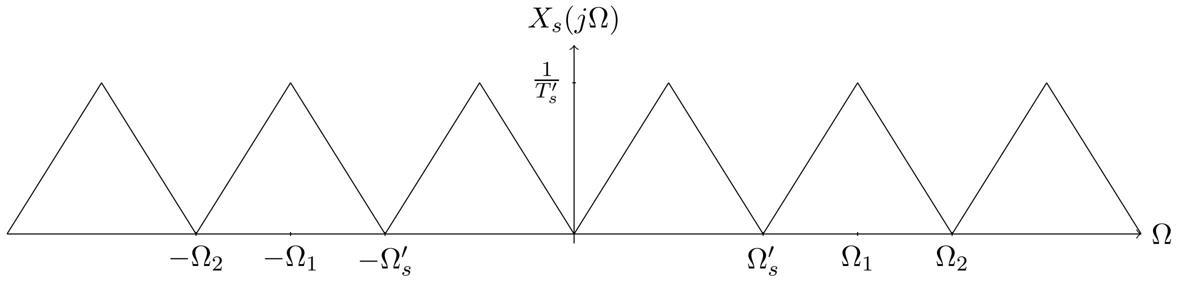


Figure 6: CTFT of $x_s(t)$.

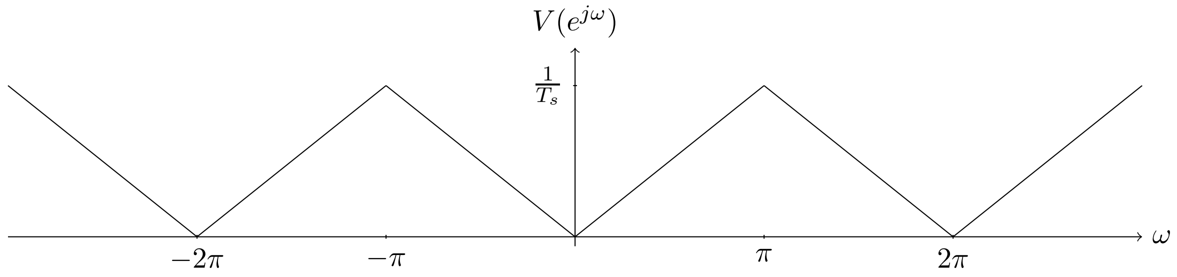


Figure 7: DTFT of $v[n]$.

(d) This has already been answered in part (c).

(e) First, it is important to know that the condition $\Omega_s \geq 2\Omega_N$ is a sufficient condition for recovering $x_c(t)$ from $x[n]$. However, it is not a **necessary** condition. Indeed, this example shows that there are cases in which reconstruction of $x_c(t)$ is possible although $\Omega_s > 2\Omega_N$.

The key observation is that even if the maximal frequency of $X_c(j\Omega)$ is equal to $\Omega_N = \Omega_2$, the actual bandwidth occupied by the signal $x_c(t)$ is smaller, because it is a band-limited signal. The actual bandwidth that it occupies is $2(\Omega_2 - \Omega_1)$. In addition, we observe that the sampling frequency f_s' is chosen such that when shifting $X_c(j\Omega)$ by multiples of Ω_s', non-zero values of the shifts never overlap.

If we look at Figure 8, we realize that by applying a passband filter to $X_s(j\Omega)$ (and some scaling), we can recover $X_c(j\Omega)$. More precisely, we see that

$$X_c(j\Omega) = H(j\Omega)X_s(j\Omega),$$

where

$$H(j\Omega) = \begin{cases} T_s' & \text{for } |\Omega| \in [\Omega_1, \Omega_2] \\ 0 & \text{otherwise.} \end{cases}$$
$$= T_s' \text{rect}\left(\frac{\Omega - \frac{\Omega_1+\Omega_2}{2}}{\Omega_2 - \Omega_1}\right) + T_s' \text{rect}\left(\frac{\Omega + \frac{\Omega_1+\Omega_2}{2}}{\Omega_2 - \Omega_1}\right)$$
$$= \frac{\pi}{\Omega_2 - \Omega_1} \text{rect}\left(\frac{\Omega - \frac{\Omega_1+\Omega_2}{2}}{\Omega_2 - \Omega_1}\right) + \frac{\pi}{\Omega_2 - \Omega_1} \text{rect}\left(\frac{\Omega + \frac{\Omega_1+\Omega_2}{2}}{\Omega_2 - \Omega_1}\right),$$

where we have used the fact that $\Omega_s' = 100 \cdot 2\pi = 2(\Omega_2 - \Omega_1)$. From the lecture notes, we know that the inverse Fourier transform of the rectangular function is

$$\frac{2\pi}{A} \text{rect}\left(\frac{\Omega}{A}\right) \overset{\mathcal{F}^{-1}}{\Rightarrow} \text{sinc}\left(\frac{At}{2\pi}\right).$$

Using this fact and the shift property of the CTFT, we obtain

$$h(t) = \frac{1}{2} \text{sinc}\left(\frac{(\Omega_2 - \Omega_1)t}{2\pi}\right)\left(e^{j\frac{\Omega_1+\Omega_2}{2}t} + e^{-j\frac{\Omega_1+\Omega_2}{2}t}\right)$$
$$= \frac{1}{2} \text{sinc}\left(\frac{t}{2T_s'}\right)\left(e^{j\frac{\Omega_1+\Omega_2}{2}t} + e^{-j\frac{\Omega_1+\Omega_2}{2}t}\right).$$

Finally, because multiplication in frequency is convolution in time, we can recover $x_c(t)$ as

$$x_c(t) = h(t) * x_s(t),$$

where we use the continuous-time convolution.

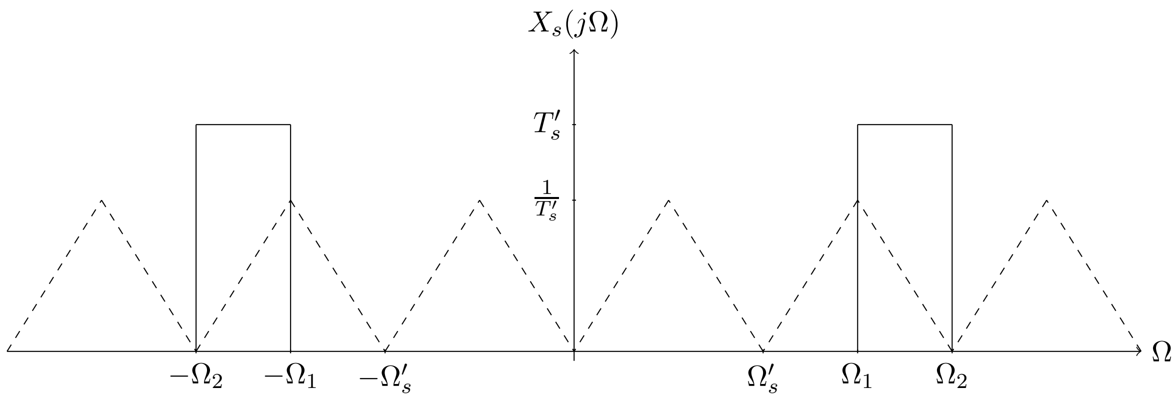


Figure 8: CTFT of $x_s(t)$.

Problem 4

(a) The code for the functions `Izero` and `Ifirst` is given on Figures 9a and 9b. Figure 10 shows the corresponding plot.

(b) See Figure 10.

```
function f = Izero(t)

f = zeros(size(t));
f(abs(t) <= .5) = 1;

end
```

(a) `Izero.m`: Implements zero-order interpolation.

```
function f = Ifirst(t)

f = zeros(size(t));

f(abs(t) < 1) = 1 - abs(t(abs(t) < 1));

end
```

(b) `Ifirst.m`: Implements first-order interpolation.

Figure 9: Code for Problem 4(a).

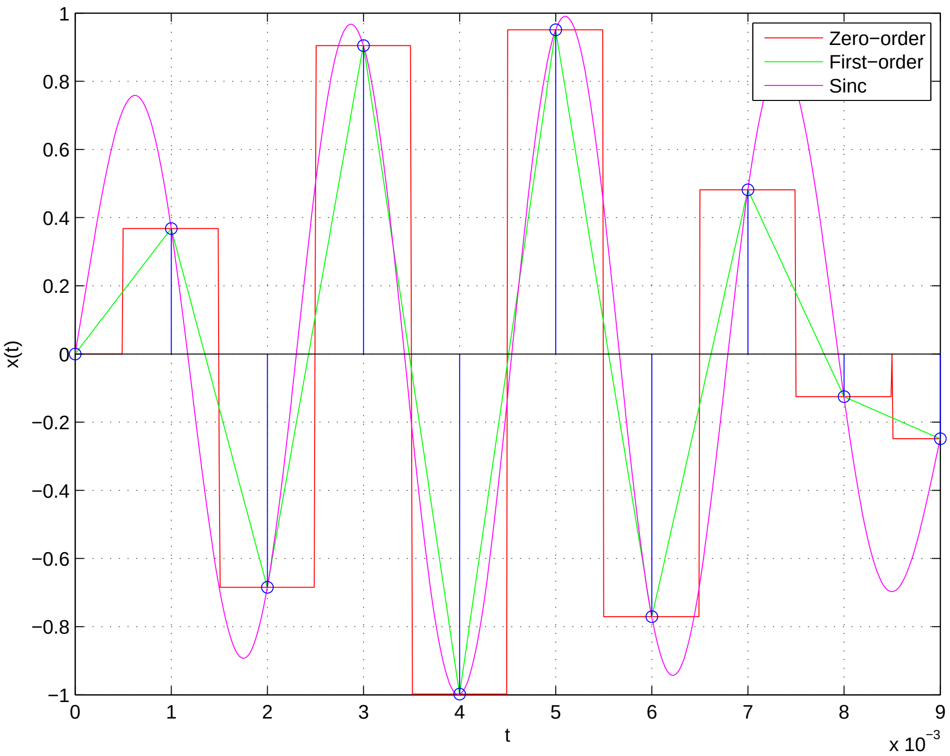


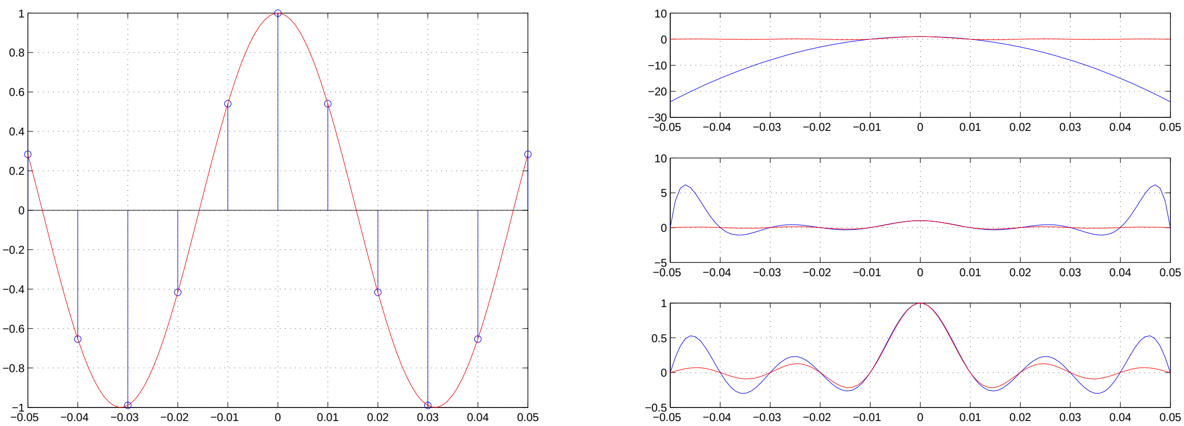
Figure 10: Interpolation plot for Problem 4(a).

```
function l = lagrange(t, n, N, Ts)

l = ones(size(t));

for k = -N:N
    if k ~= n
        l = l .* (t/Ts - k) ./ (n - k);
    end
end
```

Figure 11: `lagrange.m`: Implementation of Lagrange interpolation.



(a) Lagrange interpolation for sinusoidal signal.

(b) For large N, the Lagrange polynomial approximates the sinc function.

Figure 12: Lagrange interpolation.

(c) The code for the Lagrange interpolation is shown in Figure 11.

(d) The plot of the Lagrange interpolation is shown on Figure 12a.

(e) The plot of $\mathrm{sinc}(t/T_s)$ and $L_0^{(N)}$ is shown in Figure 12b.