Solutions: Homework Set # 1

Mathematical Prerequisites

Problem 1 (Geometric Series)

(a) We can write

$$\begin{aligned}\sum_{k=0}^{n} kr^k &= r\sum_{k=0}^{n}\frac{\mathrm{d}}{\mathrm{d}r}r^k \\ &= r\frac{\mathrm{d}}{\mathrm{d}r}\frac{1-r^{n+1}}{1-r}.\end{aligned}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1-r^{n+1}}{1-r^n} = \frac{-(n+1)r^n + nr^{n+1} + 1}{(1-r)^2}.$$

Multiplying with r gives the final result:

$$\sum_{k=0}^{n} kr^k = \frac{-(n+1)r^{n+1} + nr^{n+2} + r}{(1-r)^2}.$$

(b) Using the general formula for geometric series, we have

$$\sum_{k=0}^{m} e^{j\frac{2\pi}{n}k} = \frac{1-e^{j\frac{2\pi}{n}(m+1)}}{1-e^{j\frac{2\pi}{n}}}.$$

If $m = ln - 1$, then this is equal to

$$\frac{1-e^{j2\pi l}}{1-e^{j\frac{2\pi}{n}}} = 0.$$

(c) We have

$$\begin{aligned}\sum_{k=0}^{\infty} t[k] &= \sum_{k=0}^{\infty}\frac{1}{4^k} + \sum_{k=0}^{\infty}\left(\frac{1}{3j}\right)^k \\ &= \frac{1}{1-1/4} + \frac{1}{1-1/3j} \\ &= \frac{4}{3} + \frac{9-3j}{10} \\ &= \frac{67}{30} - \frac{3}{10}j.\end{aligned}$$

(d) We have

$$\begin{aligned}\sum_{k=0}^{n-1} e^{j\theta k} &= \frac{1-e^{j\theta n}}{1-e^{j\theta}} \\ &= \frac{e^{j\theta n/2}\left(e^{-j\theta n/2}-e^{j\theta n/2}\right)}{e^{j\theta/2}\left(e^{-j\theta/2}-e^{j\theta/2}\right)} \\ &= e^{j\theta(n-1)/2}\frac{\sin(\theta n/2)}{\sin(\theta/2)}.\end{aligned}$$

Taking the norm gives

$$\left|\frac{\sin(\theta n/2)}{\sin(\theta/2)}\right|.$$

Problem 2 (Complex Numbers)

(a) $|e^z| = |e^{x+jy}| = |e^x e^{jy}| = |e^x||e^{jy}| = |e^x| = e^x$.

(b) Using $\sin z = (e^{jz} - e^{-jz})/2j$ and $\cos z = (e^{jz} + e^{-jz})/2$, we get

$$\begin{aligned}\sin z &= \frac{e^{jx}e^{-y} - e^{-jx}e^{y}}{2j} \\ &= \frac{e^{j(x-\pi/2)}e^{-y} - e^{-j(x+\pi/2)}e^{y}}{2}\end{aligned}$$

and so

$$\begin{aligned}\Re \sin z &= \frac{1}{2}\left[e^{-y}\cos(x-\pi/2) - e^{y}\cos(x+\pi/2)\right] \\ &= \frac{1}{2}\left(e^{-y}\sin x + e^{y}\sin x\right) \\ &= \sin x \cosh y,\end{aligned}$$

and

$$\begin{aligned}\Im \sin z &= \frac{1}{2}\left[e^{-y}\sin(x-\pi/2) + e^{y}\sin(x+\pi/2)\right] \\ &= \frac{1}{2}\left(-e^{-y}\cos x + e^{y}\cos x\right) \\ &= \cos x \sinh y.\end{aligned}$$

Similarly, we get $\Re \cos z = \cos x \cosh y$ and $\Im \cos z = -\sin x \sinh y$.

(c) We want to find a and b such that $e^{a+jb} = z$. First, we have $e^a = |z|$, so $a = \log|z|$. Next, note that $\arg e^{jb} = \arg z$ for all b such that $b = \arg z + 2\pi k$, $k \in \mathbb{Z}$. Thus, $\log z = \log|z| + j(\arg z + 2\pi k)$, $k \in \mathbb{Z}$. (Properly speaking, $\log z$ is not a function, but the reverse image of the complex exponential.)

Problem 3 (Linear Algebra)

(a) It is easiest to start with the column with the most zeroes. Then

$$|\mathbf{A}| = (-1)^{3+2} \cdot 2 \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & 0 \end{vmatrix} + (-1)^{4+2}(-3) \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix},$$

where

$$\begin{aligned} \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & 0 \end{vmatrix} &= (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} + (-1)^{3+2} \cdot 2 \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} \\ &= 3 - 2 \cdot 4 = -5 \end{aligned}$$

and

$$\begin{aligned} \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} &= (-1)^{3+3} \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \\ &= 5, \end{aligned}$$

so

$$\begin{aligned} |\mathbf{A}| &= (-2) \cdot (-5) - 3 \cdot 5 \\ &= 10 - 15 = -5. \end{aligned}$$

(b) Setting $\lambda = 0$ yields $\det \mathbf{A} = \lambda_1 \cdots \lambda_n$.

(c) Expanding the term $(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$, one sees that the coefficient of $(-\lambda)^{n-1}$ is $\lambda_1 + \cdots + \lambda_n$. Now consider the equation for the determinant of a matrix $\mathbf{B}$,

$$\det(\mathbf{B}) = \sum_{j=1}^{n} B_{i,j} (-1)^{i+j} \det\left(\mathbf{B}^{\backslash(i,j)}\right),$$

for some $i = 1, \ldots, n$. For any term of this sum where $i \neq j$ (i.e., involving a non-diagonal element of $\mathbf{B}$), $\mathbf{B}^{\backslash(i,j)}$ will miss exactly *two* diagonal elements of the original matrix $\mathbf{B}$. Applied to the matrix $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$, this means that any term involving a non-diagonal element can contain λ at most with power $n-2$. Therefore, the only term involving $(-\lambda)^{n-1}$ in $\det(\mathbf{A} - \lambda \mathbf{I})$ comes from the diagonal product $(a_{11} - \lambda) \cdots (a_{nn} - \lambda)$, in which the coefficient of $(-\lambda)^{n-1}$ is $a_{11} + \cdots + a_{nn}$. Since the coefficient of $(-\lambda)^{n-1}$ is unique, we get the desired result.

Introduction to MATLAB

Problem 4 (MATLAB)

Nothing to hand in for this problem.

Problem 5 (Operators)

Suppose we have a matrix A given by

$$\begin{bmatrix} a+ib & c+id \\ e+if & g+ih \end{bmatrix}. \tag{1}$$

What will be the MATLAB output of the following commands: i) `A'` ii) `A.'`, iii) `fliplr(A)`, iv) `sum(A)`, v) `sum(A,2)`, vi) `A*A`, vii) `A.*A`. Hint: Have a look in the documentation for *arithmetic operations*.

`C = A*B` is the linear algebraic product of the matrices A and B.
`A.*B` is the element-by-element product of the arrays A and B. A and B must have the same size, unless one of them is a scalar.
`A'` is the linear algebraic transpose of A. For complex matrices, this is the complex conjugate transpose.
`A.'` is the array transpose of A. For complex matrices, this does not involve conjugation.
`Sum` returns sums along different dimensions of an array.
`Sum(A,b)` sums along the dimension of A specified by scalar dim.
`B = fliplr(A)` returns A with columns flipped in the left-right direction, that is, about a vertical axis. (Similarly, `flipud` flips its argument in the up-down direction.)

Problem 6

The n-th coefficient of $c(x)$ is given by

$$c_n = \sum_{k=0}^{\max(N-1,M-1)} a_k b_{n-k}.$$

This corresponds exactly to the convolution of the two sequences **a** and **b**, so the right MATLAB command is

```
>> c = conv(a, b)
```

Problem 7 (Sequences)

(a)
```
n=1:45;
a=sin(2*pi*n/15);
stem(n,a);
```

The output is shown on Figure 1.

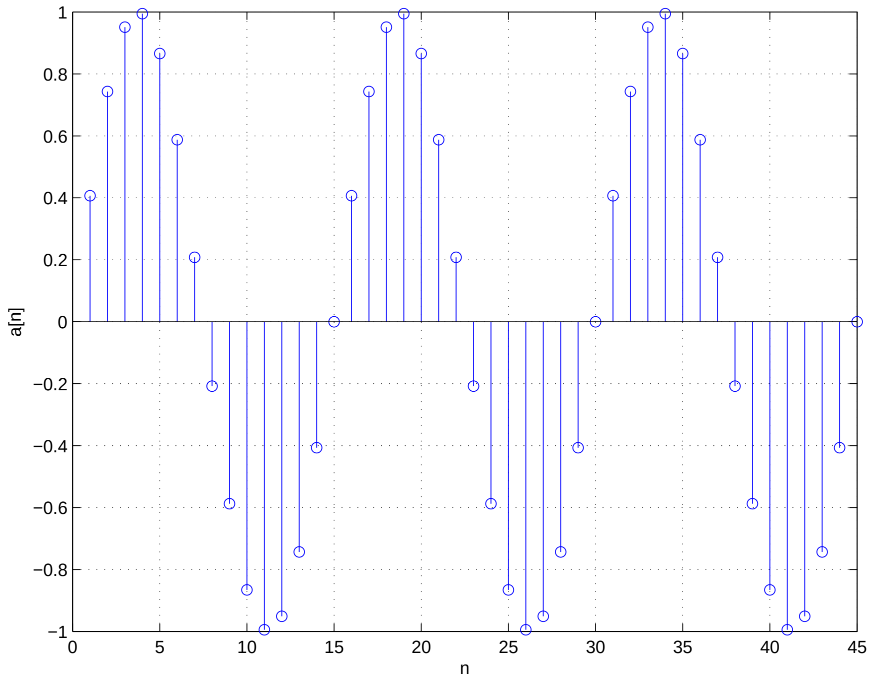

Figure 1: Plot output of Problem 7

(b)
```
b=a(1:5:end)
stem(b);
```

(c)
```
function output=cshiftright(a,N)
size_a=length(a);
output=[a(N:end) a(1:size_a-N)];
```

(d)
```
c=cshiftright(a,8)
stem(c)
```

Problem 8 (Audio)

(b)
```
>> stem(data(1:100));
```

(c)
```
len = length(data);
soundsc(data(len:-1:1,fs));
stem(data(100:-1:1);
```

(d)
```
n=1:10e3;
a=sin(440 .* n ./ 2e3);
c=sin(523.25 .* n ./ 2e3);
e=sin(659.26 .* n ./ 2e3);
soundsc(a,2e3)
soundsc(a+e,2e3)
```

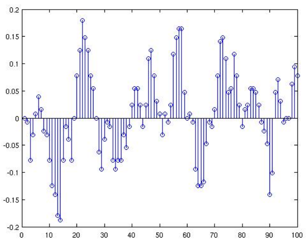

Figure 2: Problem 8 – Part (b)

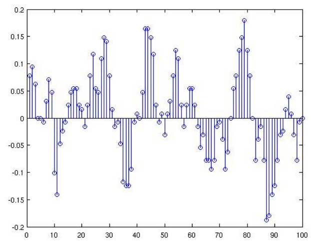

Figure 3: Problem 8 – Part (c)

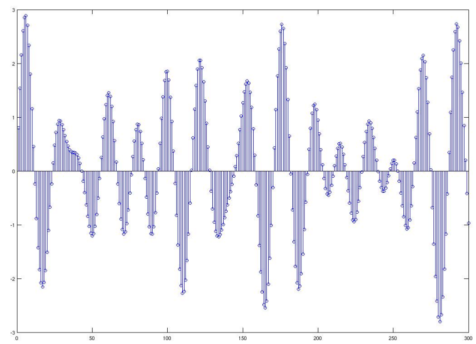

Figure 4: Problem 8 – Part (f)

```
soundsc(a+e+c,2e3)
d=a+e+c;
b=a+e;
stem(d(1:300))
soundsc(a(10e3:-1:1),2e3)
soundsc(b(10e3:-1:1),2e3)
soundsc(d(10e3:-1:1),2e3)
```

Problem 9 (Images)

(a)
```
A=imread('lena.jpg');
colormap(gray(256));
imagesc(A)
```

(b)
```
B=diag(A);
stem(B)
```

(c)
```
soundsc(A(1:256*256));
```

(d)
```
[a,fs] = wavread('handel.wav');
```

Figure 5: Problem 9 – Part (a)

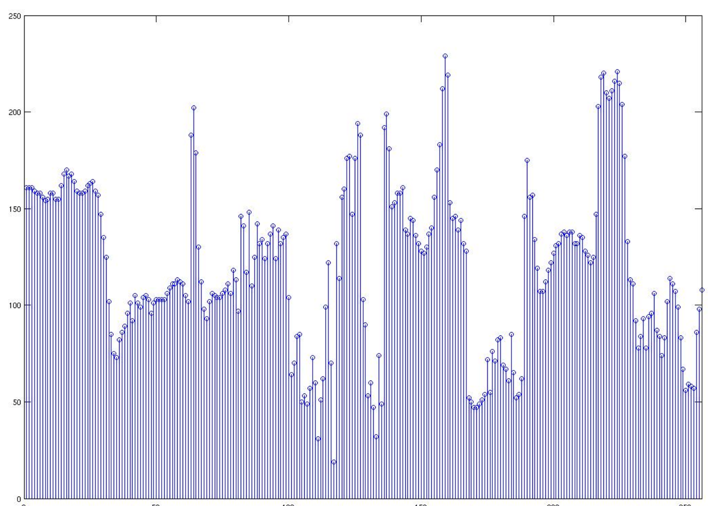

Figure 6: Problem 9 – Part (b)

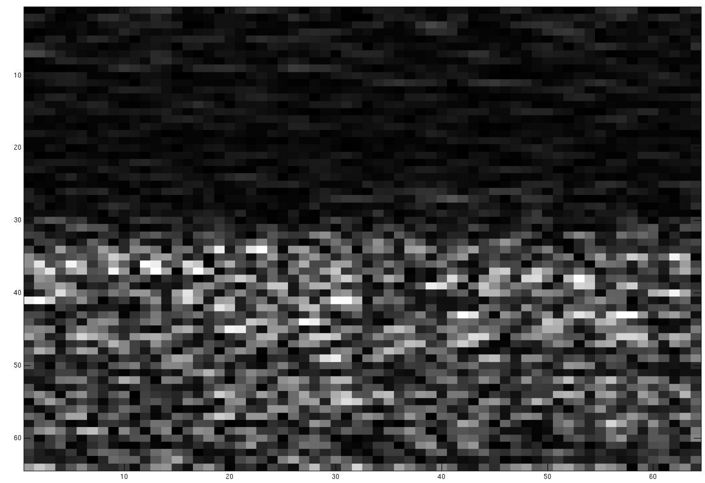

Figure 7: Problem 8 - Part 4

```
pic=abs(reshape(data(1:4096),64,64))';
pic=pic./max(max(pic)).*256;
imagesc(pic)
colormap(gray(256));
```