

Solutions to Evaluation Test

Problem 1 (Complex numbers)

(a) The polar representation of a complex number $z = x + jy$ is given by the following equations:

$$\rho = \sqrt{x^2 + y^2}$$
$$\varphi = \arctan\left(\frac{y}{x}\right)$$

Therefore we have:

$$z = \rho e^{j\varphi}$$
$$z = \sqrt{61} e^{j \arctan(\frac{5}{-6})}$$

(b) We know that ω can be written as:

$$\begin{aligned}\omega &= \rho(\cos(\varphi) + j\sin(\varphi)) \\ &= \frac{3}{4}\left(\cos(-\frac{\pi}{4}) + j\sin(-\frac{\pi}{4})\right) \\ &= \frac{3}{4}\left(\cos(-\frac{\pi}{4}) - j\sin(\frac{\pi}{4})\right) \\ &= \frac{3}{4}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) \\ &= \frac{3}{4\sqrt{2}} + j\frac{-3}{4\sqrt{2}}\end{aligned}$$

(c) We can see that

$$z^* = \rho e^{-j\varphi}$$

Therefore:

$$\begin{aligned}\left|\frac{z}{z^*}\right| &= \left|\frac{\rho e^{j\varphi}}{\rho e^{-j\varphi}}\right| \\ &= \left|\frac{e^{j\varphi}}{e^{-j\varphi}}\right| \\ &= \left|e^{j2\varphi}\right| \\ &= 1\end{aligned}$$

(d) Take

$$x + iy = \sqrt{-1 + j\sqrt{3}}$$

Squaring on both side we get

$$(x^2 - y^2) + j2xy = -1 + j\sqrt{3}$$

We therefore have the following set of equations:

$$\begin{aligned} x^2 - y^2 &= -1 \\ 2xy &= \sqrt{3} \\ \implies y &= \frac{\sqrt{3}}{2x} \\ \implies y^2 &= \frac{3}{4x^2} \end{aligned}$$

We replace this in the first equation:

$$\begin{aligned} x^2 - \frac{3}{4x^2} &= -1 \\ \implies 4x^4 - 3 &= -4x^2 \\ \implies 4x^4 + 4x^2 - 3 &= 0 \end{aligned}$$

This can be written as:

$$(2x^2 + 3)(2x^2 - 1) = 0$$

We can solve this to get the following values for x^2:

$$\begin{aligned} x^2 &= \frac{-3}{2} \\ \implies x &= \pm j\sqrt{\frac{3}{2}} \end{aligned}$$

Or,

$$\begin{aligned} x^2 &= \frac{1}{2} \\ \implies x &= \pm\sqrt{\frac{1}{2}} \end{aligned}$$

Now, if $x = \pm j\sqrt{\frac{3}{2}}$

$$\begin{aligned} y &= \pm j\frac{\sqrt{3}}{2\sqrt{\frac{3}{2}}} \\ \implies y &= \pm j\sqrt{\frac{1}{2}} \\ \implies x + jy &= \pm\left(\sqrt{\frac{1}{2}} + j\sqrt{\frac{3}{2}}\right) \end{aligned}$$

Now, if $x = \pm\sqrt{\frac{1}{2}}$

$$\begin{aligned} y &= \pm\frac{\sqrt{3}}{2\sqrt{\frac{1}{2}}} \\ \implies y &= \pm\sqrt{\frac{3}{2}} \\ \implies x + jy &= \pm\left(\sqrt{\frac{1}{2}} + j\sqrt{\frac{3}{2}}\right) \end{aligned}$$

We therefore realize that the solution to our initial equation is

$$\sqrt{-1 + j\sqrt{3}} = \pm\left(\sqrt{\frac{1}{2}} + j\sqrt{\frac{3}{2}}\right)$$

(e)

(f) We can write j as:

$$\begin{aligned} j &= e^{j\frac{\pi}{2}} \\ \implies ln(j) &= ln(e^{j\frac{\pi}{2}}) \\ &= j\frac{\pi}{2} \\ \implies ln(j) &= j\frac{\pi}{2} \end{aligned}$$

Problem 2 (Polynomials)

(a) We can write $x^3 + 64$ as:

$$(x^3 + 64) = (x + 4)(x^2 - 4x + 16)$$

Solving the binomial equation we get:

$$x = 2 \pm j2\sqrt{3}$$

Therefore the solutions are:

$$\begin{aligned} x &= -4 \\ x &= 2 + j2\sqrt{3} \\ x &= 2 - j2\sqrt{3} \end{aligned}$$

(b) Let us first substitute x^3 by y. We have:

$$\begin{aligned} y^2 - 3y - 4 &= 0 \\ \implies (y-4)(y+1) &= 0 \\ \implies y &= 4, -1 \\ \implies x^3 &= 4, -1 \end{aligned}$$

We now solve $x^3 = 4$:

$$\begin{aligned} x^3 &= 4e^{j2\pi k}(k = 0, 1, 2) \\ \implies x &= 4^{\frac{1}{3}} e^{\frac{j2\pi k}{3}}(k = 0, 1, 2 \end{aligned}$$

We solve the other equation:

$$\begin{aligned} x^3 &= e^{j\pi(2k+1)}(k = 0, 1, 2) \\ \implies x &= e^{\frac{j\pi(2k+1)}{3}}(k = 0, 1, 2) \end{aligned}$$

Problem 3 (Series)

(a) $e^{j\frac{\pi}{3}}$ does not depend on k, hence:

$$\sum_{k=0}^{8} e^{j\frac{\pi}{3}} = 9e^{j\frac{\pi}{3}}$$

(b) We have

$$\begin{aligned} &\sum_{k=-2}^{+\infty} \frac{52^{k+3}}{(-3)^k} \\ = \quad &52^3 \left(\frac{52^{-2}}{(-3)^2} + \frac{52^{-1}}{(-3)^{-1}} + \sum_{k=0}^{\infty} \frac{52^{k+3}}{(-3)^k} \right) \end{aligned}$$

Now,

$$\begin{aligned} \left| \frac{52}{-3} \right| &> 1 \\ \implies \sum_{k=0}^{+\infty} \frac{52^{k+3}}{(-3)^k} & \end{aligned}$$

does not converge. This means that the sum that we are trying to calculate does not converge.

(c) We have the following:

$$\begin{aligned} & \sum_{k=0}^{+\infty} \frac{1}{3+7n} \\ \geq & \sum_{k=0}^{+\infty} \frac{1}{7+7n} \\ = & \sum_{k=0}^{+\infty} \frac{1}{7} \frac{1}{1+n} \\ = & \frac{1}{7} \sum_{k=1}^{+\infty} \frac{1}{n} \\ = & \infty \\ \Longrightarrow \sum_{k=0}^{+\infty} \frac{1}{3+7n} = & +\infty \end{aligned}$$

(d) Use the following formulae:

$$\begin{aligned} \sum_{k=0}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=0}^{n} k &= \frac{n(n+1)}{2} \end{aligned}$$

and we get:

$$\sum_{k=0}^{n} (k^2 - k) = \frac{n(n-1)(n+1)}{3}$$

(e) We remark that:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Therefore:

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

(f) The idea is to split it into two sums:

$$\begin{aligned}
& \sum_{m=0}^{+\infty} \frac{m+1}{m!} \\
= & \sum_{m=0}^{+\infty} \frac{m}{m!} + \sum_{m=0}^{+\infty} \frac{1}{m!} \\
= & \sum_{m=1}^{+\infty} \frac{m}{m!} + \sum_{m=0}^{+\infty} \frac{1}{m!} \\
= & \sum_{m=1}^{+\infty} \frac{1}{(m-1)!} + \sum_{m=0}^{+\infty} \frac{1}{m!} \\
= & \sum_{l=0}^{+\infty} \frac{1}{l!} + \sum_{m=0}^{+\infty} \frac{1}{m!} \\
= & e^1 + e^1 \\
= & 2e
\end{aligned}$$

Problem 4 (Linear algebra)

(a) We can calculate $B^*B + C$ as:

- B^* being the conjugate transpose of B, it is of size $3X2$. This means that B^*B can be performed and it results in a $3X3$ matrix.
- Both B^*B and C are $3X3$ matrices and hence the addition is possible.

(b) To multiply two matrices F and G, if F is a mXn matrix, then G has to be a nXp matrix for some positive integers m,n and p. Here this is clearly not the case.

(c) $\left|DD^T\right|$ is well defined.

(d) DD^* is a $2X2$ matrix whereas B is a $2X3$ matrix, hence addition is not possible.

(e) I is a $2X2$ matrix whereas C is a $3X3$ matrix, hence this is not possible.

Problem 5 (Integrals)

(a) The idea is to use the following formula and replace in the integral:

$$\cos(at) = \frac{e^{jat} + e^{-jat}}{2}$$

Therefore the integral becomes:

$$
\begin{aligned}
\int_0^{2\pi} \cos\left(\frac{t}{2}\right) e^{j\frac{t}{3}}\, dt &= \int_0^{2\pi} \left(\left(\frac{e^{jat}+e^{-jat}}{2}\right) e^{j\frac{t}{3}}\right)\, dt \\
&= \frac{1}{2}\left\{\int_0^{2\pi} e^{j\frac{t}{2}} e^{j\frac{t}{3}}\, dt + \int_0^{2\pi} e^{j\frac{-t}{2}} e^{j\frac{t}{3}}\, dt\right\} \\
&= \frac{1}{2}\left\{\int_0^{2\pi} e^{j\frac{5t}{6}}\, dt + \int_0^{2\pi} e^{j\frac{-t}{6}}\, dt\right\} \\
&= \frac{1}{2}\left\{\frac{-6j}{5}\left(e^{j\frac{5\pi}{3}}-1\right) + 6j\left(e^{-j\frac{\pi}{3}}-1\right)\right\} \\
&= \frac{1}{2}\left\{\frac{6j}{5}\left(e^{j\frac{\pi}{3}}+1\right) + 6j\left(e^{-j\frac{\pi}{3}}-1\right)\right\} \\
&= \ldots
\end{aligned}
$$

(b) Using the hint we have:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \sin(\log_4(t^\pi))\delta(t-2)\, dt &= \sin(\log_4(2^\pi)) \\
&= \sin\left(\log_4\left(4^{\frac{\pi}{2}}\right)\right) \\
&= \sin\left(\frac{\pi}{2}\right) \\
&= 1
\end{aligned}
$$