Homework Set # 8

Problem 1 (Order of Up- and Downsampling)

We consider the two systems given in Figure 1.

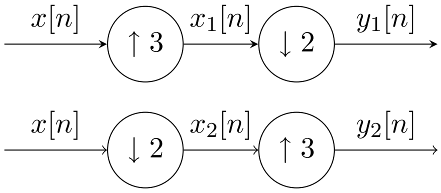

Figure 1: Systems for Problems 1 (a)–(e).

(a) For $X\left(e^{j\omega}\right)$ as given in Figure 2, sketch $X_1\left(e^{j\omega}\right)$ and $Y_1\left(e^{j\omega}\right)$.

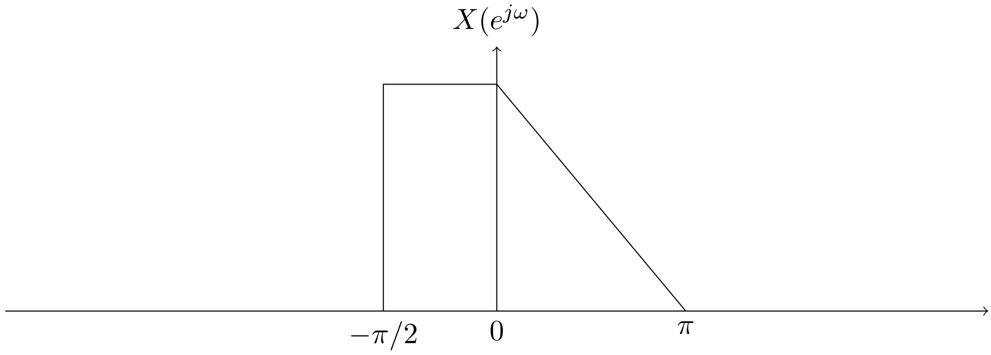

Figure 2: $X\left(e^{j\omega}\right)$ for Exercise 1

(b) Again for $X\left(e^{j\omega}\right)$ as given in Figure 2, sketch $X_2\left(e^{j\omega}\right)$ and $Y_2\left(e^{j\omega}\right)$.

(c) What is the relationship between $Y_1\left(e^{j\omega}\right)$ and $Y_2\left(e^{j\omega}\right)$?

For parts $(d),(e)$ we consider a general spectrum $X\left(e^{j\omega}\right)$ and *not* specific to Figure 2. Therefore, the answers would be in terms of a general spectrum $X\left(e^{j\omega}\right)$.

(d) For general $X\left(e^{j\omega}\right)$, write both $Y_1\left(e^{j\omega}\right)$ and $Y_2\left(e^{j\omega}\right)$ in terms of $X\left(e^{j\omega}\right)$. (*Hint: Use the properties of the Fourier transform of upsampled and downsampled signals as done in class.*)

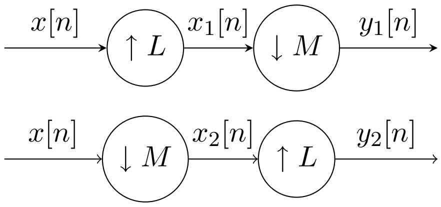

Figure 3: Systems for Problem 1 (f).

(e) Using the expressions you have obtained in part (*d*) prove the relationship between $Y_1\left(e^{j\omega}\right)$ and $Y_2\left(e^{j\omega}\right)$.

Hint: Generalize the relationship discovered in part (c) to an arbitrary $X\left(e^{j\omega}\right)$.

(f) Consider now the two general up-/downsampling systems in Figure 3, with upsampling factor L and downsampling factor M. In class you have seen that the Z-Transforms of $y_1[n]$ and $y_2[n]$ are given by

$$Y_1(z) = \sum_{k=0}^{M-1} X(W_M^k z^{L/M})$$

and

$$Y_2(z) = \sum_{k=0}^{M-1} X(W_M^{kL} z^{L/M}).$$

In this last part you will show that the above two Z-Transforms $Y_1(z)$ and $Y_2(z)$ are equal if and only if L and M are coprime, *i.e.*, if they do not have any prime factors in common. First note that the sums for $Y_1(z)$ and $Y_2(z)$ differ only in the terms W_M^k and W_M^{kL}. Therefore $Y_1(z)$ and $Y_2(z)$ are equal if and only if the sequence $\{W_M^{kL}\}_{k=0}^{M-1}$ is a permutation of the sequence $\{W_M^k\}_{k=0}^{M-1}$, *i.e.*, it contains exactly the same elements but possibly in a different order.

1. Show that if L and M have a common factor Q, then there exists $k \in \{1, \ldots, M-1\}$ such that $W_M^{kL} = 1$, and conclude that in this case $Y_1(z) \neq Y_2(z)$.
   Hint: $W_M^k = e^{-j2\pi k/M} = W_M^{k \bmod M}$ for all $k \in \mathbb{Z}$.
2. Assume now that L and M are coprime. Using Bezout's identity (given below), show that for every $l \in \{0, \ldots, M-1\}$ there exists a $k \in \{0, \ldots, M-1\}$ such that $kL \bmod M = l$.

   Bezout's identity: If a and b are nonzero integers with greatest common divisor d, then there exist integers x and y such that $ax + by = d$. (If a and b are coprime, then $d = 1$.)
3. Conclude that the operations of upsampling and downsampling commute if and only if L and M are coprime.

Problem 2

Consider the system shown in Fig. 4, where $u[n] = U_L(x[n])$, $v[n] = u[n-N]$, and $y[n] = D_M(v[n])$.

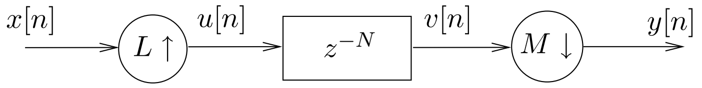

Figure 4: system in Problem. 2.

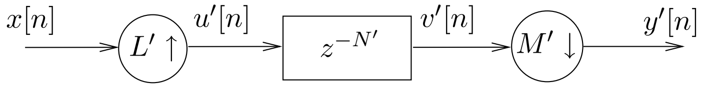

Figure 5: equivalent system in Problem. 2.

(a) Write the z transform of each of the sequences $u[n]$, $v[n]$, and $y[n]$ in terms of $X(z)$, the z-transform of the input sequence.

(b) Let $d = \text{g.c.d}(L, M)$ be the largest common divisor of L and M. Assume $d \mid N$ (*i.e.*, there exist some integer N' such that $N = N'd$), and $L = L'd$ and $M = M'd$. Show that the above system is equivalent to the system shown in Fig. 5 (*i.e.*, $Y(z) = Y'(z)$)
Hint: The following two summations are equal

$$\sum_{k=0}^{M'd-1} f(k) = \sum_{p=0}^{d-1} \sum_{q=0}^{M'-1} f(pM' + q).$$

(c) Show that if $d \nmid N$ (*i.e.*, d is not a divisor of N), then $Y(z) = 0$.

Problem 3 (Hierarchical Filterbanks with Infinite Length Filters)

Consider the tree-structured analysis filter bank shown in Fig. 6. Assume that $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ are the ideal low-pass and high-pass filters, respectively, *i.e.*,

$$H_0(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi, \end{cases} \qquad H_1(e^{j\omega}) = \begin{cases} 0 & |\omega| \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < |\omega| \le \pi. \end{cases}$$

(a) Draw the DTFT of each of the sequences $y_0[n]$, $y_1[n]$, and $y_2[n]$, for a given input $x[n]$, for which the DTFT is given as in Fig. 7.

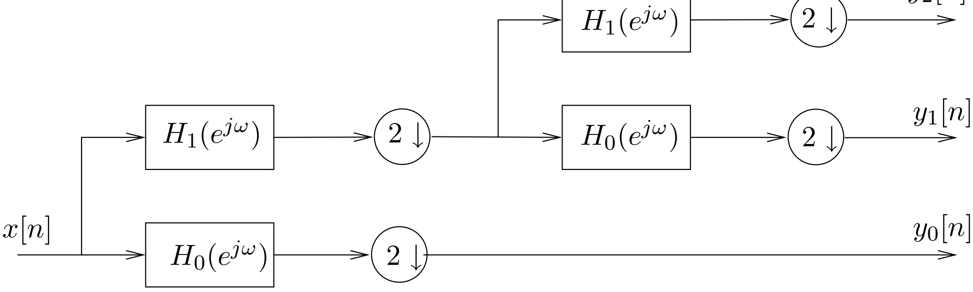

Figure 6: Tree-structured Analysis filter bank

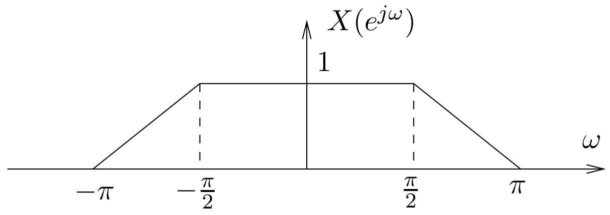

Figure 7: Spectrum of the input $x[n]$.

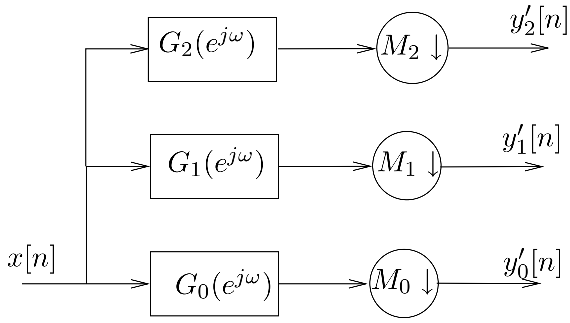

Figure 8: Equivalent Analysis filter bank

(b) Find the filters $G_0(e^{j\omega})$, $G_1(e^{j\omega})$, $G_2(e^{j\omega})$, and the down-sampling factors M_1, M_2, and M_3, such that the system shown in Fig. 8 be equivalent to the system in Fig. 6 (*i.e.,* for arbitrary input $x[n]$, they give $y'_i[n] = y_i[n]$, for $i = 1, 2, 3$).
Hint: Use the properties of down-sampling and in particular the Nobel identity.

(c) Consider the corresponding synthesis filter bank given in Fig. 9. Find the corresponding filters $F_0(e^{j\omega})$ and $F_1(e^{j\omega})$ such that this filter bank can reconstruct the original signal, *i.e.,* $x[n] = x'[n]$.

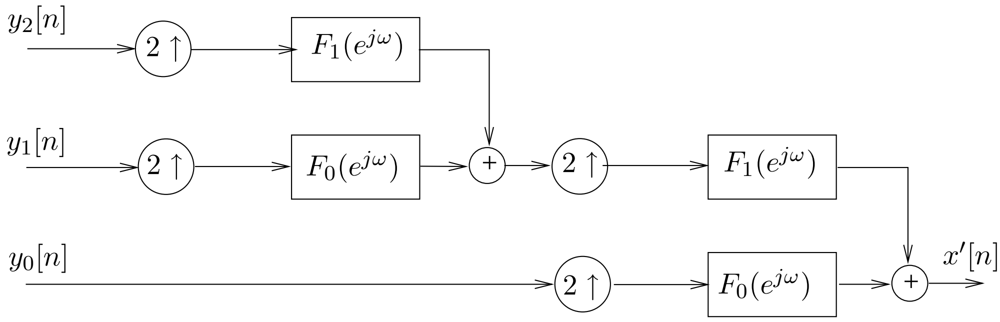

Figure 9: Tree-structured synthesis filter bank

Problem 4 (Block Discrete-Time Fourier Transform)

Consider the following method to represent a length-L sequence $x[n]$. We segment $x[n]$ into Q subsequences $x^{(i)}[n]$ of length N such that

$$x^{(i)}[n] = \begin{cases} x[n] & n = iN + l, \quad l = 0, 1, \ldots, N-1, \\ 0 & \text{otherwise}, \end{cases}$$

$i = 0, \ldots, Q-1$, and take the DFT of each subsequence independently,

$$X^{(i)}[k] = \sum_{l=0}^{N-1} x^{(i)}[iN + l] e^{-j2\pi kl/N} \quad k = 0, 1, \ldots, N-1.$$

This is called a *block discrete-time Fourier Transform* or block DFT, since the signal is divided into blocks of size N, which are then Fourier transformed. You have already encountered this transform in class as the spectrogram.

(a) Find $\varphi_k^{(i)}[n]$ such that we have the reconstruction formula

$$x[n] = \sum_{i=0}^{Q-1} \sum_{k=0}^{N-1} X^{(i)}[k] \varphi_k^{(i)}[n].$$

(b) Show that the sequences $\varphi_k^{(i)}[n]$ form an orthogonal set. These sequences are the *basis functions* of the block discrete-time Fourier Transform.

(c) Show that for $N = 2$, the basis functions $\varphi_k^{(i)}[n]$ are scaled versions of the Haar basis functions seen in class.

Note also that for $Q = 1$, $N = L$, the block DFT is equivalent to the DFT. Therefore, the Haar decomposition and the DFT can be seen as special cases of a more general transform. In the Haar decomposition, the basis functions have very short support in time, and therefore large support in frequency. In the DFT, the basis functions have infinite support in time and a single point of support in frequency (every DFT basis vector represents a single frequency).

The block DFT is an instance of a wider class of transforms, called *wavelet transforms.* Wavelet transforms are characterized by the projection of a signal onto a set of linearly independent vectors; the linear independence of the vectors guarantees the invertibility of the transform. Thus, any set of linearly independent vectors can be chosen as a basis for a wavelet transform, the choice of the basis depending on the particular properties of the signal that one wants to study.

(d) Consider the filterbank shown on Figure 10. This filterbank implements the block discrete-time Fourier Transform such that $v_k[n] = X^{(n)}[k]$. Determine the filters $h_k[n]$ and $g_k[n]$, $k = 0, \ldots, N-1$ such that $\hat{x}[n] = x[n]$, *i.e.*, for perfect reconstruction, and justify your results.

Note also that as opposed to the perfect reconstruction filterbank seen in class, which used perfect lowpass and highpass filters, the filters of this filterbank have overlapping spectra. Ideal filters are therefore not a necessary condition for a perfect reconstruction filterbank.

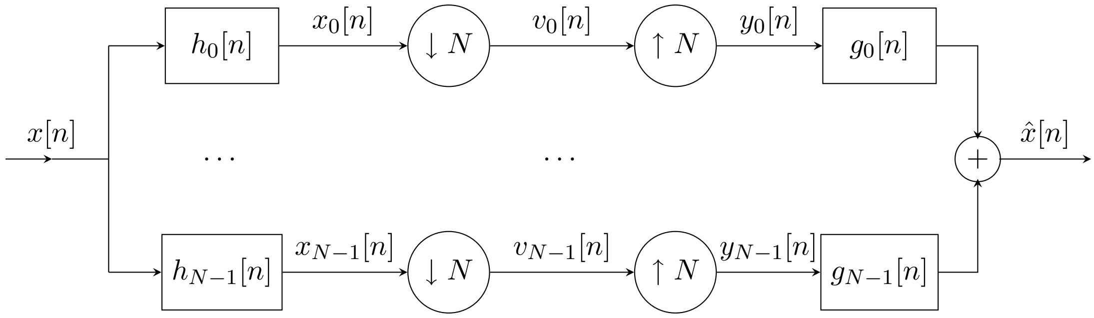

Figure 10: Filterbank for Problem 4.

Problem 5 (Sampling Rate Change and Aliasing)

In this exercise, we change the sampling rate of an audio file. We do this operation twice, first without and then with the antialiasing filters, and we compare the two outcomes.

(a) Download the file "vista.wav" from the course webpage. Load it into Matlab and listen to it. (Use the functions *wavread* and *soundsc*.) What is the sampling frequency of "vista.wav"?

(b) Now, use the functions *downsample* and *upsample* to change the sampling rate to 13230 Hz. Do the operations without using the antialiasing filter. Plot the magnitude of the DFT of the outcome. Listen to the resulting audio sequence. Comment on the quality.

(c) Now, use the functions *downsample*, *upsample*, *firpm* and *conv* to change the sampling rate to 13230 Hz. For the appropriate antialiasing, use 81-point Parks-McClellan filters with a transition band of width 0.04π. Plot the magnitude of the DFT of the outcome. Listen to the resulting audio sequence. Comment on the quality.

(d) Compare the audio sequences obtained with and without antialiasing filters. What is the effect of aliasing? Explain what happened in both cases, and how the result makes the two sequences sound different.

(e) Now, use the functions *decimate* and *interp* to repeat question (c). What would you expect? Can you notice a difference in the (audio) outcomes of questions (c) and (e)?