Problem 1

(a) Yes, it can be BIBO stable. For stability, the unit circle has to lie in the ROC, i.e.,  
r_{\text{min}} < 1 < r_{\text{max}}.

(b) The system \( y[n] = \alpha^{-n}x[n] \) is linear, because if \( x[n] = ax_1[n] + bx_2[n] \), then  
\[ v[n] = \alpha^{-n}(ax_1[n] + bx_2[n]) = a\alpha^{-n}x_1[n] + b\alpha^{-n}x_2[n]. \]

However, the system is not time-invariant, because if we define \( \tilde{x}[n] = x[n - n_0] \), then  
\[ \tilde{v}[n] = \alpha^{-n}\tilde{x}[n] = \alpha^{-n}x[n - n_0] \neq v[n - n_0]. \]

(c) Since the overall system is the interconnection of 3 linear systems, it is also linear (by the linearity of the convolution). To find out whether it is time-invariant, we derive the impulse response:

\[
\begin{align*}
v[n] &= \alpha^{-n}x[n] \\
w[n] &= (\alpha^{-n}x[n]) * h[n] \\
y[n] &= \alpha^{n}((\alpha^{-n}x[n]) * h[n]) \\
    &= \alpha^{n}\left(\sum_{k=-\infty}^{\infty} \alpha^{-k}x[k]h[n-k]\right) \\
    &= \sum_{k=-\infty}^{\infty} \alpha^{n-k}x[k]h[n-k] \\
    &= x[n] * (\alpha^{n}h[n]).
\end{align*}
\]

Hence, we see that the overall impulse response is \( \alpha^{n}h[n] \). Now, we see that the overall system is actually time-invariant, because if \( \tilde{x}[n] = x[n - n_0] \), then  
\[
\tilde{x}[n] * (\alpha^{n}h[n]) = \sum_{k=-\infty}^{\infty} \tilde{x}[n-k]\alpha^{k}h[k] \\
    = \sum_{k=-\infty}^{\infty} x[n-k-n_0]\alpha^{k}h[k] \\
    = \sum_{k=-\infty}^{\infty} x[(n-n_0)-k]\alpha^{k}h[k] \\
    = (x * (\alpha^{n}h))[n-n_0].
\]

(d) If \( H(z) \) has a ROC that is the ring \( r_{\text{min}} < |z| < r_{\text{max}} \), then we know that \( H(z) \) has at least one pole \( p_{\text{low},1} \) with absolute value \( |p_{\text{low},1}| = r_{\text{min}} \), at least one pole \( p_{\text{high},1} \) such that
\[ |p_{\text{high,1}}| = r_{\text{max}}, \text{ and that there are no poles with absolute value in } (r_{\text{min}}, r_{\text{min}}). \text{ Hence, we can write} \]

\[
h[n] = \sum_{i=1}^{N_{\text{low}}} (p_{\text{low},i})^n u[n] + \sum_{k=1}^{N_{\text{high}}} (p_{\text{high},k})^n u[-n - 1] + \text{additional terms},
\]

where we assumed that there are \( N_{\text{low}} \) poles \( p_{\text{low},i} \) with \(|p_{\text{low},i}| = r_{\text{min}}\) (lying on the smaller circle) and that there are \( N_{\text{high}} \) poles \( p_{\text{high},k} \) with \(|p_{\text{high},k}| = r_{\text{max}}\) (lying on the larger circle). The additional terms indicated correspond to poles that are located away from the ROC.

Now,

\[
\alpha^n h[n] = \sum_{i=1}^{N_{\text{low}}} (p_{\text{low},i}\alpha)^n u[n] + \sum_{k=1}^{N_{\text{high}}} (p_{\text{high},k}\alpha)^n u[-n - 1] + \alpha^n \text{additional terms},
\]

and one can see that the ROC will now be ring \( |\alpha|r_{\text{min}} < |z| < |\alpha|r_{\text{max}} \). Hence, the system is stable if \(|\alpha|r_{\text{min}} < 1 < |\alpha|r_{\text{max}}\).

**Problem 2**

[DFT and z-Transform]

\[
X(z) = \sum_{n=0}^{N-1} x[n] z^{-n} = \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\omega n \frac{2\pi}{N}} \right] z^{-n} 
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{j\omega n \frac{2\pi}{N}} z^{-n} 
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \frac{1 - z^{-N}}{1 - e^{j\omega \frac{2\pi}{N}} z^{-1}} 
\]

\[
= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X[k]}{1 - e^{j\omega \frac{2\pi}{N}} z^{-1}} 
\]

**Problem 3**

[Minimum Phase System]

(a-i) Since \(|c| > 1\), the transfer function has a zero outside the unit circle and it is not a minimum phase system. In order to make it minimum phase we add a zero at \( z = 1/c^* \) instead of \( z = c \) and then compensate it in the all-pass filter:

\[
H(z) = \frac{1 - cz^{-1}}{1 - dz^{-1}} 
\]

\[
= \frac{z^{-1} - c^*}{1 - dz^{-1}} \frac{1 - cz^{-1}}{z^{-1} - c^*} 
\]

\[
= \frac{z^{-1} - c^*}{1 - dz^{-1}} \cdot \left( \frac{c}{c^*} \right) \frac{z^{-1} - \frac{1}{c^*}}{1 - \left( \frac{1}{c^*} \right)^{-1}} 
\]

\(|z| = 1/|c^*| < 1\)

\(|p| = |d| < 1\)  

causal all-pass filter
So we have
\[ H_{\text{min}}(z) = \frac{z^{-1} - e^s}{1 - dz^{-1}} \]
and
\[ H_{\text{ap}}(z) = \frac{c}{e^s} \frac{z^{-1} - \frac{1}{c}}{1 - \left(\frac{1}{c}\right)^{z^{-1}}} = \frac{1 - cz^{-1}}{z^{-1} - e^s} \]
By plug in \( e^{j\omega} \) we have
\[ H_{\text{ap}}(e^{j\omega}) = \frac{1 - ce^{-j\omega}}{e^{-j\omega} - e^s} = \frac{1 - |c|e^{j\theta}e^{-j\omega}}{e^{-j\omega} - |c|e^{-j\theta}} \]
The group-delay is the negative derivation of the phase. The phase of the transfer function can be computed as the following:

\[
\arg H_{\text{ap}}(e^{j\omega}) = \arg \left( \frac{1 - |c|e^{j\theta}e^{-j\omega}}{e^{-j\omega} - |c|e^{-j\theta}} \right) = \arg \left( \frac{e^{j\omega} - 1 - |c|e^{j\theta}e^{-j\omega}}{1 - |c|e^{-j\theta}e^{j\omega}} \right) = \arg \left[ e^{j\omega} \right] + \arg \left[ 1 - |c|e^{j\theta}e^{-j\omega} \right] - \arg \left[ 1 - |c|e^{-j\theta}e^{j\omega} \right] = \omega + \arg \left[ (1 - |c| \cos(\omega - \theta)) + i(|c| \sin(\omega - \theta)) \right] - \arg \left[ (1 - |c| \cos(\omega - \theta)) + i(-|c| \sin(\omega - \theta)) \right] = \omega + \tan^{-1} \frac{|c| \sin(\omega - \theta)}{1 - |c| \cos(\omega - \theta)} - \tan^{-1} \frac{-|c| \sin(\omega - \theta)}{1 - |c| \cos(\omega - \theta)} = \omega + 2 \tan^{-1} \frac{|c| \sin(\omega - \theta)}{1 - |c| \cos(\omega - \theta)}
\]
Note that \( \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \). Therefore
\[
grd H_{\text{ap}}(e^{j\omega}) = -\frac{d}{d\omega} \arg H_{\text{ap}}(e^{j\omega}) = -\frac{d}{d\omega} \omega - 2 \tan^{-1} \frac{|c| \sin(\omega - \theta)}{1 - |c| \cos(\omega - \theta)} = -\frac{2}{1 - |c| \cos(\omega - \theta)^2 + (|c| \sin(\omega - \theta))^2} \cdot \frac{|c|^2 + |c| \cos(\omega - \theta)}{1 - |c| \cos(\omega - \theta)^2} = -\frac{|c|^2 - 1}{1 + |c|^2 - 2|c| \cos(\omega - \theta)} > 0
\]
where the inequality follows from \( |c| > 1 \) and holds for any \( \omega \).
(a-ii) Since the group delay of any all-pass system is positive, we have
\[
grd [H(z)] = \text{grd} [H_{\text{min}}(z)] + \text{grd} [H_{\text{ap}}(z)] > \text{grd} [H_{\text{min}}(z)]
\]
which proves that the minimum phase system has the minimum group-delay among all the systems with the same frequency response.
(b-i) From the causality of the systems, we have \( h_{\min}[n] = h[n] = 0 \) for \( n < 0 \). Using the Parseval’s theorem we can write

\[
\sum_{n=0}^{\infty} |h_{\min}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{\min}(e^{j\omega})|^2 d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega \\
= \sum_{n=0}^{\infty} |h[n]|^2
\]

where \((a)\) follows form \( |H(z)| = |H_{\min}(z)| \).

(b-ii) By the definition of \( H(z) \), it should be of the form

\[
H(z) = cQ(z)(1 - \frac{1}{\alpha^* z^{-1}})
\]

where the constant \( c \) should be determined such that \( |H(z)| = |H_{\min}(z)| \), which yields \( |c| = |\alpha| \). Therefore \( H(z) = |\alpha|Q(c)(1 - \frac{1}{\alpha^* z^{-1}}) \)

(b-iii)

\[
H_{\min}(z) = Q(z)(1 - \alpha z^{-1}) \\
\Rightarrow h_{\min}[n] = q[n] \ast (\delta[n] - \alpha\delta[n - 1]) \\
= q[n] - \alpha q[n - 1]
\]

and

\[
H(z) = |\alpha|Q(z)(1 - \frac{1}{\alpha^* z^{-1}}) \\
\Rightarrow h_{\min}[n] = q[n] \ast \left( |\alpha|\delta[n] - \frac{|\alpha|}{\alpha^*}\delta[n - 1] \right) \\
= |\alpha|q[n] - \frac{|\alpha|}{\alpha^*}q[n - 1]
\]

(b-iv) We can write

\[
D_m = \sum_{n=0}^{m} |h_{\min}[n]|^2 - \sum_{n=0}^{m} |h[n]|^2 \\
= \sum_{n=0}^{m} |q[n] - \alpha q[n - 1]|^2 - \sum_{n=0}^{m} |||\alpha|q[n] - \frac{|\alpha|}{\alpha^*}q[n - 1]||^2 \\
= \sum_{n=0}^{m} [||q[n]|^2 + |\alpha|^2|q[n - 1]|^2 - 2\Re\{\alpha q[n - 1]\}q[n]] \\
- \sum_{n=0}^{m} \left[ |\alpha|^2|q[n]|^2 + \frac{|\alpha|^2}{|\alpha^*|^2}|q[n - 1]|^2 - 2\Re\left\{ \frac{|\alpha|^2}{|\alpha^*|^2} q[n - 1]q[n] \right\} \right] \\
= \sum_{n=0}^{m} [[|q[n]|^2 - |q[n - 1]|^2] - |\alpha|^2 \sum_{n=0}^{m} [|q[n]|^2 - |q[n - 1]|^2] \\
= (1 - |\alpha|^2)||q[n]|^2 (b) > 0
\]

where \((a)\) and \((b)\) follows from the causality of \( q[n] \) and the fact \( |\alpha| < 1 \), respectively.

(b-v) We have seen in part (b-iv) that \( \sum_{n=0}^{m} |h_{\min}[n]|^2 > \sum_{n=0}^{m} |h[n]|^2 \). This means although \( h_{\min}[n] \) and \( h[n] \) have the same total energy, the energy of \( h_{\min}[n] \) will be appear earlier than the energy of \( h[n] \) and the minimum phase system has the minimum energy-delay among all the systems with the same magnitude response.
Problem 4

1. Let \( H(z) = \sum_n h[n]z^{-n} \). We have that
\[
\frac{d}{dz} H(z) = \frac{d}{dz} \left( \sum_n h[n]z^{-n} \right)
= \sum_n (-n)h[n]z^{-n-1}
= -z^{-1} \sum_n nh[n]z^{-n}
\]
and the relation follows directly.

2. We have that
\[
\alpha^n u[n] \leftrightarrow \frac{Z}{1 - \alpha z^{-1}}.
\]
Using (a) we find
\[
n\alpha^n u[n] \leftrightarrow \frac{-z \frac{d}{dz} \left( \frac{1}{1 - \alpha z^{-1}} \right)}{(1 - \alpha z^{-1})^2} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}.
\]
Thus,
\[
(n+1)\alpha^{n+1} u[n+1] \leftrightarrow \frac{Z}{1 - \alpha z^{-1}} \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}
\]
and
\[
(n+1)\alpha^n u[n+1] \leftrightarrow \frac{Z}{(1 - \alpha z^{-1})^2} \frac{1}{(1 - \alpha z^{-1})^2}.
\]
The relation follows by noticing that
\[
(n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]
\]
since when \( n = -1 \) both sides are equal to zero.

3. The system is causal since the ROC corresponds to the outside of a circle of radius \( \alpha \) (or equivalently since the impulse response is zero when \( n < 0 \)). The system is stable when the unit circle lies inside the ROC, i.e. when \( |\alpha| \leq 1 \).

4. When \( \alpha = 0.8 \), the angular frequency of the pole is \( \omega = 0 \). Thus the filter is lowpass. When \( \alpha = -0.8 \), \( \omega = \pi \) and the filter is highpass.

Problem 5

1. The transfer function of the system is given by:
\[
Y(z)(1 - 3.25z^{-1} + 0.75z^{-2}) = X(z)(z^{-1} + 3z^{-2})
\]
\[
H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 3z^{-2}}{1 - 3.25z^{-1} + 0.75z^{-2}} = \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})} = \frac{z + 3}{(z - 0.25)(z - 3)}
\]
Since the system is causal, the convergence region is \( |z| > 3 \). We can see that there is the pole \( z = 3 \) that is out of the unit circle and therefore the system is unstable (Figure 1).

\[
>> zplane ([0 1 3], [1 -3.25 0.75])
\]
2. Z-transform of the output signal is:

\[ Y(z) = H(z)X(z) = \frac{z^{-1}(1 + 3z^{-1})}{(1 - 0.25z^{-1})(1 - 3z^{-1})} = \frac{z^{-1} + 3z^{-2}}{1 - 0.25z^{-1}}. \]

From \( Y(z) \) we can see that the unstable pole \( z = 3 \) is canceled and only the pole \( z = 0.25 \) of \( Y(z) \) is left. Since the system is causal, even from the unstable system we can get the stable output if the unstable pole is canceled by the input signal.

3. ```
>> x=[1 -3 2 -1 zeros(1,25)];
>> y=filter([0 1 3],[1 -3.25 0.5],x);
subplot(211), stem(x), title ('input signal x[n]') subplot(212), stem(y), title('output signal y[n]')
```

![Figure 2: Solution 4 (c).](image)

On Figure 2 we can see that the unstable pole is not canceled and the output signal is therefore \( Y(z) \) is unstable function.

**Problem 6**

1. We have clearly:

\[ X(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} = H(z^2) + z^{-1}G(z^2) \]
2. The ROC is determined by the zeros of the transform. Since the sequence is two sided, the ROC is a ring bounded by two poles $z_L$ and $z_R$ such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$. Consider $H(z)$; if $z_0$ is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$. Therefore the ROC for $H(z^2)$ is the ring $|z_L| < |z| < |z_R|$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so

$$
\text{ROC} = 0.8 < |z| < 2.
$$