Chapter 8, The Z-Transform: Problem Solutions

Problem 1

(a) Yes, it can be BIBO stable. For stability, the unit circle has to lie in the ROC, *i.e.*, $r_{\min} < 1 < r_{\max}$.

(b) The system $y[n] = \alpha^{-n}x[n]$ is linear, because if $x[n] = ax_1[n] + bx_2[n]$, then

$$v[n] = \alpha^{-n}(ax_1[n] + bx_2[n]) = a\alpha^{-n}x_1[n] + b\alpha^{-n}x_2[n].$$

However, the system is not time-invariant, because if we define $\tilde{x}[n] = x[n - n_0]$, then

$$\tilde{v}[n] = \alpha^{-n}\tilde{x}[n] = \alpha^{-n}x[n - n_0] \neq v[n - n_0].$$

(c) Since the overall system is the interconnection of 3 linear systems, it is also linear (by the linearity of the convolution). To find out whether it is time-invariant, we derive the impulse response:

$$\begin{aligned}
v[n] &= \alpha^{-n}x[n] \\
w[n] &= \big(\alpha^{-n}x[n]\big) * h[n] \\
y[n] &= \alpha^{n}\left(\big(\alpha^{-n}x[n]\big) * h[n]\right) \\
&= \alpha^{n}\left(\sum_{k=-\infty}^{\infty} \alpha^{-k}x[k]h[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty} \alpha^{n-k}x[k]h[n-k] \\
&= x[n] * (\alpha^{n}h[n]) .
\end{aligned}$$

Hence, we see that the overall impulse response is $\alpha^n h[n]$. Now, we see that the overall system is actuall time-invariant, because if $\tilde{x}[n] = x[n - n_0]$, then

$$\begin{aligned}
\tilde{x}[n] * (\alpha^{n}h[n]) &= \sum_{k=-\infty}^{\infty} \tilde{x}[n-k]\alpha^{k}h[k] \\
&= \sum_{k=-\infty}^{\infty} x[n-k-n_0]\alpha^{k}h[k] \\
&= \sum_{k=-\infty}^{\infty} x[(n-n_0)-k]\alpha^{k}h[k] \\
&= \left(x * (\alpha^{\cdot} h)\right)[n-n_0].
\end{aligned}$$

(d) If $H(z)$ has a ROC that is the ring $r_{\min} < |z| < r_{\max}$, then we know that $H(z)$ has at least one pole $p_{\text{low},1}$ with absolute value $|p_{\text{low},1}| = r_{\min}$, at least one pole $p_{\text{high},1}$ such that

$|p_{\text{high},1}| = r_{\text{max}}$, and that there are no poles with absolute value in $(r_{\text{min}}, r_{\text{min}})$. Hence, we can write

$$h[n] = \sum_{i=1}^{N_{\text{low}}} (p_{\text{low},i})^n u[n] + \sum_{k=1}^{N_{\text{high}}} (p_{\text{high},k})^n u[-n-1] + \text{additional terms},$$

where we assumed that there are N_{low} poles $p_{\text{low},i}$ with $|p_{\text{low},i}| = r_{\text{min}}$ (lying on the smaller circle) and that there are N_{high} poles $p_{\text{high},k}$ with $|p_{\text{high},k}| = r_{\text{max}}$ (lying on the larger circle). The additional terms indicated correspond to poles that are located away from the ROC.

Now,

$$\alpha^n h[n] = \sum_{i=1}^{N_{\text{low}}} (p_{\text{low},i}\alpha)^n u[n] + \sum_{k=1}^{N_{\text{high}}} (p_{\text{high},k}\alpha)^n u[-n-1] + \alpha^n \text{additional terms},$$

and one can see that the ROC will now be ring $|\alpha| r_{\text{min}} < |z| < |\alpha| r_{\text{max}}$. Hence, the system is stable if $|\alpha| r_{\text{min}} < 1 < |\alpha| r_{\text{max}}$.

Problem 2

[DFT AND z-TRANSFORM]

$$\begin{aligned}
X(z) &= \sum_{n=0}^{N-1} x[n] z^{-n} = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\omega n \frac{2\pi}{N}} \right] z^{-n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \left[X[k] \sum_{n=0}^{N-1} e^{j\omega n \frac{2\pi}{N}} z^{-n} \right] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \frac{1 - z^{-N}}{1 - e^{j\omega \frac{2\pi}{N}} z^{-1}} \\
&= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X[k]}{1 - e^{j\omega \frac{2\pi}{N}} z^{-1}}
\end{aligned}$$

Problem 3

[MIMIMUM PHASE SYSTEM]

(a-i) Since $|c| > 1$, the transfer function has a zero outside the unit circle and it is not a minimum phase system. In order to make it minimum phase we add a zero at $z = 1/c^*$ instead of $z = c$ and then compensate it in the all-pass filter:

$$\begin{aligned}
H(z) &= \frac{1 - cz^{-1}}{1 - dz^{-1}} \\
&= \frac{z^{-1} - c^*}{1 - dz^{-1}} \cdot \frac{1 - cz^{-1}}{z^{-1} - c^*} \\
&= \underbrace{\frac{z^{-1} - c^*}{1 - dz^{-1}}}_{\substack{|z| = 1/|c^*| < 1 \\ |p| = |d| < 1}} \cdot \underbrace{\left(\frac{c}{c^*}\right) \frac{z^{-1} - \frac{1}{c}}{1 - \left(\frac{1}{c}\right)^* z^{-1}}}_{\text{causal all-pass filter}}
\end{aligned}$$

So we have

$$H_{\min}(z) = \frac{z^{-1} - c^*}{1 - dz^{-1}}$$

and

$$H_{ap}(z) = \frac{c}{c^*} \frac{z^{-1} - \frac{1}{c}}{1 - \left(\frac{1}{c}\right)^* z^{-1}} = \frac{1 - cz^{-1}}{z^{-1} - c^*}$$

By plug in $e^{i\omega}$ we have

$$H_{ap}(e^{i\omega}) = \frac{1 - ce^{-i\omega}}{e^{-i\omega} - c^*} = \frac{1 - |c|e^{i\theta}e^{-i\omega}}{e^{-i\omega} - |c|e^{-i\theta}}$$

The group-delay is the negative derivation of the phase. The phase of the transfer function can be computed as the following:

$$\begin{aligned}
\arg H_{ap}(e^{i\omega}) &= \arg \left(\frac{1 - |c|e^{i\theta}e^{-i\omega}}{e^{-i\omega} - |c|e^{-i\theta}} \right) \\
&= \arg \left(e^{i\omega} \frac{1 - |c|e^{i\theta}e^{-i\omega}}{1 - |c|e^{-i\theta}e^{i\omega}} \right) \\
&= \arg \left[e^{i\omega} \right] + \arg \left[1 - |c|e^{i\theta}e^{-i\omega} \right] - \arg \left[1 - |c|e^{-i\theta}e^{i\omega}) \right] \\
&= \omega + \arg \left[(1 - |c|\cos(\omega - \theta)) + i(|c|\sin(\omega - \theta)) \right] \\
&\quad - \arg \left[(1 - |c|\cos(\omega - \theta)) + i(-|c|\sin(\omega - \theta)) \right] \\
&= \omega + \tan^{-1} \frac{|c|\sin(\omega - \theta)}{1 - |c|\cos(\omega - \theta)} - \tan^{-1} \frac{-|c|\sin(\omega - \theta)}{1 - |c|\cos(\omega - \theta)} \\
&= \omega + 2\tan^{-1} \frac{|c|\sin(\omega - \theta)}{1 - |c|\cos(\omega - \theta)}
\end{aligned}$$

Note that $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$. Therefore

$$\begin{aligned}
\mathrm{grd} H_{ap}(e^{i\omega}) &= -\frac{d}{d\omega} \arg H_{ap}(e^{i\omega}) \\
&= -\frac{d}{d\omega}\omega - \frac{d}{d\omega} 2\tan^{-1} \frac{|c|\sin(\omega - \theta)}{1 - |c|\cos(\omega - \theta)} \\
&= -1 - 2\frac{(1 - |c|\cos(\omega - \theta))^2}{(1 - |c|\cos(\omega - \theta))^2 + (|c|\sin(\omega - \theta))^2} \cdot \frac{-|c|^2 + |c|\cos(\omega - \theta)}{(1 - |c|\cos(\omega - \theta))^2} \\
&= -1 - \frac{-|c|^2 + |c|\cos(\omega - \theta)}{1 + |c|^2 - 2|c|\cos(\omega - \theta)} \\
&= \frac{|c|^2 - 1}{1 + |c|^2 - 2|c|\cos(\omega - \theta)} > 0
\end{aligned}$$

where the inequality follows from $|c| > 1$ and holds for any ω.

(a-ii) Since the group delay of any all-pass system is positive, we have

$$\begin{aligned}
\mathrm{grd}\,[H(z)] &= \mathrm{grd}\,[H_{\min}(z)] + \mathrm{grd}\,[H_{ap}(z)] \\
&> \mathrm{grd}\,[H_{\min}(z)]
\end{aligned}$$

which proves that the minimum phase system has the minimum group-delay among all the systems with the same frequency response.

(b-i) From the causality of the systems, we have $h_{min}[n] = h_[n] = 0$ for $n < 0$. Using the Parseval's theorem we can write

$$\begin{aligned} \sum_{n=0}^{\infty} |h_{\min}[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{\min}(e^{i\omega})|^2 d\omega \\ &\stackrel{(a)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{i\omega})|^2 d\omega \\ &= \sum_{n=0}^{\infty} |h[n]|^2 \end{aligned}$$

where (a) follows form $|H(z)| = |H_{\min}(z)|$.

(b-ii) By the definition of $H(z)$, it should be of the form

$$H(z) = cQ(z)(1 - \frac{1}{\alpha^*}z^{-1})$$

where the constant c should be determined such that $|H(z)| = |H_{\min}(z)|$, which yields $|c| = |\alpha|$. Therefore $H(z) = |\alpha|Q(c)(1 - \frac{1}{\alpha^*}z^{-1})$

(b-iii)

$$\begin{aligned} H_{\min}(z) &= Q(z)(1 - \alpha z^{-1}) \\ \Longrightarrow h_{\min}[n] &= q[n] * (\delta[n] - \alpha\delta[n-1]) \\ &= q[n] - \alpha q[n-1] \end{aligned}$$

and

$$\begin{aligned} H_(z) &= |\alpha|Q(z)(1 - \frac{1}{\alpha^*}z^{-1}) \\ \Longrightarrow h_{\min}[n] &= q[n] * \left(|\alpha|\delta[n] - \frac{|\alpha|}{\alpha^*}\delta[n-1]\right) \\ &= |\alpha|q[n] - \frac{|\alpha|}{\alpha^*}q[n-1] \end{aligned}$$

(b-iv) We can write

$$\begin{aligned} D_m &= \sum_{n=0}^{m} |h_{\min}[n]|^2 - \sum_{n=0}^{m} |h_[n]|^2 \\ &= \sum_{n=0}^{m} |q[n] - \alpha q[n-1]|^2 - \sum_{n=0}^{m} ||\alpha|q[n] - \frac{|\alpha|}{\alpha^*}q[n-1]|^2 \\ &= \sum_{n=0}^{m} \left[|q[n]|^2 + |\alpha|^2|q[n-1]|^2 - 2\Re\{\alpha q[n-1]q[n]\}[\right] \\ &\quad - \sum_{n=0}^{m} \left[|\alpha|^2|q[n]|^2 + \frac{|\alpha|^2}{|\alpha^*|^2}|q[n-1]|^2 - 2\Re\{\frac{|\alpha|^2}{\alpha^*}q[n-1]q[n]\}\right] \\ &= \sum_{n=0}^{m} \left[|q[n]|^2 - |q[n-1]|^2\right] - |\alpha|^2 \sum_{n=0}^{m} \left[|q[n]|^2 - |q[n-1]|^2\right] \\ &\stackrel{(a)}{=} (1 - |\alpha|^2|)|q[m]|^2 \stackrel{(b)}{>} 0 \end{aligned}$$

where (a) and (b) follows from the causality of $q[n]$ and the fact $|\alpha| < 1$, respectively.

(b-v) We have seen in part (b-iv) that $\sum_{n=0}^{m} |h_{\min}[n]|^2 > \sum_{n=0}^{m} |h[n]|^2$. This means although $h_{\min}[n]$ and $h[n]$ have the same total energy, the energy of $h_{\min}[n]$ will be appear earlier than the energy of $h[n]$ and the minimum phase system has the minimum energy-delay among all the systems with the same magnitude response.

Problem 4

1. Let $H(z) = \Sigma_n h[n] z^{-n}$. We have that

$$\begin{aligned}\frac{d}{dz}H(z) &= \frac{d}{dz}\left(\Sigma_n h[n] z^{-n}\right) \\ &= \Sigma_n (-n) h[n] z^{-n-1} \\ &= -z^{-1}\Sigma_n n h[n] z^{-n}\end{aligned}$$

and the relation follows directly.

2. We have that

$$\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}.$$

Using (a) we find

$$n\alpha^n u[n] \stackrel{Z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}.$$

Thus,

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{Z}{\longleftrightarrow} z\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

and

$$(n+1)\alpha^{n} u[n+1] \stackrel{Z}{\longleftrightarrow} \frac{1}{(1-\alpha z^{-1})^2}.$$

The relation follows by noticing that

$$(n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

since when $n = -1$ both sides are equal to zero.

3. The system is causal since the ROC corresponds to the outside of a circle of radius α (or equivalently since the impulse response is zero when $n < 0$). The system is stable when the unit circle lies inside the ROC, i.e. when $|\alpha| \leq 1$.

4. When $\alpha = 0.8$, the angular frequency of the pole is $\omega = 0$. Thus the filter is lowpass. When $\alpha = -0.8$, $\omega = \pi$ and the filter is highpass.

Problem 5

1. The transfer function of the system is given by:

$$Y(z)(1 - 3.25z^{-1} + 0.75z^{-2}) = X(z)(z^{-1} + 3z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}+3z^{-2}}{1-3.25z^{-1}+0.75z^{-2}} = \frac{z^{-1}(1+3z^{-1})}{(1-0.25z^{-1})(1-3z^{-1})} = \frac{z+3}{(z-0.25)(z-3)}$$

Since the system is causal, the convergence region is $|z| > 3$. We can see that there is the pole $z = 3$ that is out of the unit circle and therefore the system is unstable (Figure 1).

```
>> zplane ([0 1 3], [1 -3.25 0.75])
```

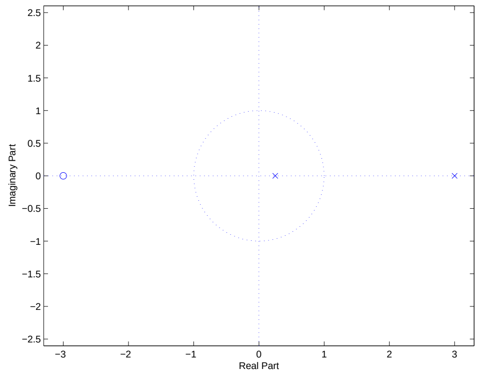

Figure 1: Pole zero plot.

2. Z-transform of the output signal is:

$$Y(z) \quad = \quad H(z)X(z) = \frac{z^{-1}(1+3z^{-1})}{(1-0.25z^{-1})(1-3z^{-1})}(1-3z^{-1}) = \frac{z^{-1}+3z^{-2}}{1-0.25z^{-1}}.$$

   From $Y(z)$ we can see that the unstable pole $z=3$ is canceled and only the pole $z=0.25$ of $Y(z)$ is left. Since the system is causal, even from the unstable system we can get the stable output if the unstable pole is canceled by the input signal.

3.
```
>> x=[1 -3 2 -1 zeros(1,25)];
>> y=filter( [0 1 3], [1 -3.25 0.5], x);
subplot(211), stem(x), title ('input signal x[n]') subplot(212),
stem(y), title('output signal y[n]')
```

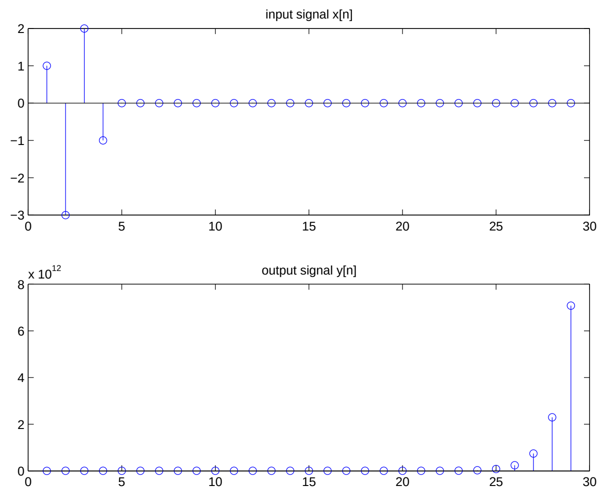

Figure 2: Solution 4 (c).

   On Figure 2 we can see that the unstable pole is not canceled and the output signal is therefore $Y(z)$ is unstable function.

Problem 6

1. We have clearly:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-2n} + g[n]z^{-(2n+1)} \\ &= H(z^2) + z^{-1}G(z^2) \end{aligned}$$

2. The ROC is determined by the zeros of the transform. Since the sequence is two sided, the ROC is a ring bounded by two poles z_L and z_R such that $|z_L| < |z_R|$ and no other pole has magnitude between $|z_L|$ and $|z_R|$. Consider $H(z)$; if z_0 is a pole of $H(z)$, $H(z^2)$ will have two poles at $\pm z^{1/2}$; however, the square root preserves the monotonicity of the magnitude and therefore no new poles will appear between the circles $|z| = \sqrt{|z_L|}$ and $|z| = \sqrt{|z_R|}$. Therefore the ROC for $H(z^2)$ is the ring $|z_L| < |z| < |z_R|$. The ROC of the sum $H(z^2) + z^{-1}G(z^2)$ is the intersection of the ROCs, and so

$$\text{ROC } = 0.8 < |z| < 2.$$