Chapter 0, Mathematical Prerequisites: Problem Solutions

Problem 1

Recall that

$$\sum_{i=0}^{k} z^k = \begin{cases} \frac{1-z^{k+1}}{1-z} & \text{for } z \neq 1 \\ N+1 & \text{for } z = 1\,. \end{cases}$$

Proof for $z \neq 0$ (for $z = 1$ is trivial)

$$s = 1 + z + z^2 + \ldots + z^N\,,$$
$$-zs = -z - z^2 - \ldots - z^N - z^{N+1}\,.$$

Summing the above two equations gives

$$(1-z)s = 1 - z^{N+1} \Rightarrow s = \frac{1-z^{N+1}}{1-z}\,.$$

Similarly

$$\sum_{k=N_1}^{N_2} z^k = z^{N_1} \sum_{k=0}^{N_2-N_1} z^k = \frac{z^{N_1} - z^{N_2} + 1}{1-z}\,.$$

1. We have

$$\sum_{n=1}^{N} s[n] = \sum_{n=1}^{N} 2^{-n} + j \sum_{n=1}^{N} 3^{-n}$$
$$= \frac{1}{2} \cdot \frac{1-2^{-N}}{1-2^{-1}} + j\frac{1}{3} \cdot \frac{1-3^{-N}}{1-3^{-1}} = (1-2^{-N}) + j\frac{1}{2}(1-3^{-N})\,.$$

Now,

$$\lim_{N\to\infty} 2^{-N} = \lim_{N\to\infty} 3^{-N} = 0\,.$$

Therefore,

$$\sum_{n=1}^{\infty} s[n] = 1 + \frac{1}{2}j\,.$$

2. We can write

$$\sum_{k=1}^{N} s[k] = \frac{j}{3} \cdot \frac{1-(j/3)^N}{1-j/3}\,.$$

Since $\left|\frac{j}{3}\right| = \frac{1}{3} < 1$, we have $\lim_{N\to\infty}(j/3)^N = 0$. Therefore,

$$\sum_{k=1}^{\infty} s[k] = \frac{1}{3j-1} = -\frac{1}{10} + j \cdot \frac{3}{10}\,.$$

3. From $z^* = z^{-1}$ with $z \in \mathbb{C}$, we have

$$zz^* = 1, \quad \forall\, z \neq 0.$$

Therefore, $|z|^2 = 1$ and, consequently, $|z| = 1$. It follows that all the z such that $z^* = z^{-1}$ describe the unit circle.

4. Remark that $e^{2k\pi} = 1$, for all $k \in \mathbb{Z}$. Therefore, $z_k = e^{\frac{2k\pi}{3}}$ is such that $z_k^3 = 1$. Now z_k is periodic of period 3, i.e. $z_k = z_{k+3l}$, for all $l \in \mathbb{Z}$. Therefore the (only) three different complex numbers are

$$z_0 = 1, \quad z_1 = e^{\frac{2\pi}{3}} \quad \text{and} \quad z_2 = e^{\frac{4k\pi}{3}}.$$

5. We have

$$\prod_{n=1}^{N} e^{j\frac{\pi}{2^n}} = e^{j\pi \sum_{n=1}^{N} 2^{-n}} = e^{j\pi \frac{1}{2} \cdot \frac{1-2^{-N}}{1-1/2}}.$$

Since $\lim_{N\to\infty} 2^{-N} = 0$,

$$\prod_{n=1}^{\infty} e^{j\frac{\pi}{2^n}} = e^{j\pi} = -1.$$

Problem 2

[GEOMETRIC SERIES]

(a)

- $$S[n+1] = S[n] + x[n+1]$$

- $$\tilde{S}[n] = rS[n] = r \cdot \sum_{k=0}^{n} a \cdot r^k = \sum_{k=1}^{n+1} a \cdot r^k = S[n+1] - a$$

- $$\begin{cases} rS[n] & = & S[n+1] - a \\ S[n+1] & = & S[n] + a \cdot r^{n+1} \end{cases}$$
$$\Longrightarrow rS[n] = S[n] + a \cdot r^{n+1} - a$$
$$\Longrightarrow S[n] = a\frac{1 - r^{n+1}}{1-r}$$

(b)

$$S = \sum_{k=0}^{\infty} x[k] = \lim_{n\to\infty} S[n] = \lim_{n\to\infty} a\frac{1-r^{n+1}}{1-r} = \frac{a}{1-r}$$

(c)

$$\begin{aligned} \sum_{k=n+1}^{m} x[k] &= \sum_{k=0}^{m} x[k] - \sum_{k=0}^{n} x[k] = S[m] - S[n] \\ &= a\frac{1-r^{m+1}}{1-r} - a\frac{1-r^{n+1}}{1-r} = a\frac{r^{n+1} - r^{m+1}}{1-r} \end{aligned}$$

Based on part(a), we have

$$\sum_{k=n+1}^{m} s[k] = \frac{a(m-n)}{1-r} + ar^{n+2}\frac{1-r^{m-n}}{1-r}$$

To be more precise, If we set $a' = \frac{ar^{n+2}}{1-r}$ we have,

$$\sum_{k=n+1}^{m} \frac{ar^{k+1}}{1-r} = \sum_{k=0}^{m-n-1} a'r^k = a'\frac{1-r^{m-n}}{1-r} = ar^{n+2}\frac{1-r^{m-n}}{(1-r)^2}$$

(d)

$$\begin{aligned} t[k] = \frac{1}{3^k} + (\frac{1}{2j})^k \Longrightarrow \sum_{k=0}^{\infty} t[k] = \sum_{k=0}^{\infty} \frac{1}{3^k} + \sum_{k=0}^{\infty} (\frac{1}{2j})^k &= \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{1}{2j}} \\ &= \frac{23-4j}{10} \end{aligned}$$

(e) Define $P = \Pi_{n=1}^{\infty} e^{j\pi/2^n}$, and consider the fact that $\ln(ab) = \ln(a) + \ln(b)$. Based on part (b) we have $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

$$\ln(P) = \sum_{n=1}^{\infty} ln(e^{j\pi/2^n}) = \sum_{n=1}^{\infty} \frac{j\pi}{2^n} = j\pi$$

So we have

$$P = e^{j\pi} = -1$$

Problem 3

[COMPLEX NUMBERS]

(a) *Remark*: If the summation over all coefficients in polynomial is zero then $x = 1$ is a root of that polynomial. If the summation over all coefficients of odd degree is equal to the summation of cooefficient of even degree (*e. g.,* the given polynomial in this problem) then $x = -1$ is a root.
Remark: In every polynomial of order n in the general form $P(x) = \sum_{k=0}^{n} a_k x^k \quad, a_n = 1$ the summation of the roots is $-a_{n-1}$. So, the summation of the roots is -2 in this problem. We can also solve the equation as the following.

$$x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1) \Longrightarrow x_1 = -1.$$

In order to solve the remaining degree 2 polynomial, we can write

$$\Delta = b^2 - 4ac = 1 - 4 = -3, x_2, x_3 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 \pm \sqrt{3}}{2}$$

In this case, the summation of the roots is $-1 + \frac{-1-\sqrt{3}}{2} + \frac{-1+\sqrt{3}}{2} = -2.$

(b) We know that $j = e^{j\frac{\pi}{2}}$

$$j^j = (e^{\frac{j\pi}{2}})^j = e^{\frac{-\pi}{2}}$$

(c) Suppose $z = re^{j\theta} \qquad , r > 0$

$$\arg(z) = |z|$$

$$\theta = r$$

$$z = \theta e^{j\theta} \qquad 0 \leq \theta \leq 2\pi$$

$$z = \theta(\cos\theta + j\sin\theta)$$

Fig. 1 illustrates all the point in the complex plane with this property.

(d) Characterize the set of complex numbers satisfying $z^* = z^{-1}$. Suppose $z = re^{j\theta} \qquad , r > 0$

$$\begin{aligned} z^* &= z^{-1} \\ re^{-j\theta} &= \frac{1}{r}e^{-j\theta} \\ r &= \frac{1}{r} \\ r = 1 &\qquad \theta \in [0, 2\pi] \end{aligned}$$

These numbers form a circle in the complex plane shown in Fig. 2.

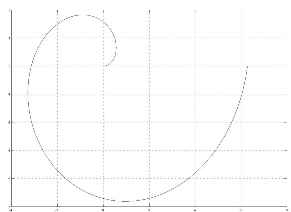

Figure 1: Problem 2(c): $z = \theta e^{j\theta}$

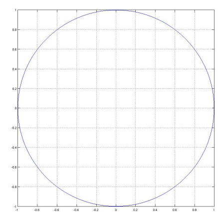

Figure 2: Problem 2(d): $z = e^{j\theta}$

Problem 4

[LINEAR ALGEBRA]

(a) Compute the determinant for the following matrix.

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & -3 & 2 & 0 \end{bmatrix}$$

Since the second column of A has only one non-sero element, it is easier to expand the determinant with respect to this column.

$$\det(A) = (-1)^{4+2}(-3)\det \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} =$$

By expanding with respect to the first column of the remaining matrix, we have

$$\begin{aligned} \det(A) &= (-3)\left[(-1)^{1+1}(2)\det \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} + (-1)^{2+1}(1)\det \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\right] \\ &= (-3)[(+1)(2)(2-2) + (-1)(1)(-1-0)] = -3 \end{aligned}$$

(b) Consider the matrices

$$B = \begin{bmatrix} j & -1 & 4 \\ 0 & 2-3j & 1 \\ -1 & 2j & 0 \\ 3 & 0 & 4-j \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & j & 1 \\ 1-5j & 1 & 4j & 2+2j \\ 1 & 3-j & 0 & -7 \end{bmatrix}.$$

Which of the following operations are well-defined (Note that you do NOT have to compute)? $C + B$, $C \cdot A^{-1}$, $B \cdot C$, $A - C$, $B + B^T$, $A + A^T$, $C^{-1} \cdot B^{-1}$, $C^* + B$,

Remark: Suppose matrice A is $n \times m$ and matrice B is $p \times q$. $A+B$ and $A-B$ are well-defined if and only if $p = n$ and $q = m$. $A \cdot B$ is well defined if and only if $m = p$. A^{-1} exists if and only if $n = m$ and $\det(A) \neq 0$. A^T and A^* are $m \times n$ matrices.
So, the following matrices are well-defined. $C \cdot A^{-1}$, $B \cdot C$, $A + A^T$, $C^* + B$,

(c) Let $x = [1,\ 2j,\ 1+j,\ 0]$. Compute Ax^T and xB. $Ax^T = [1-j, 3+2j, 2+2j, 1-4j]^T$
$xB = [-1, 3+6j, 4+2j]$

(d) Compute the determinant of $D = xx^T$ and $E = x^Tx$.

$$E = [7]$$

$$D = \begin{bmatrix} 1 & 0-2.j & 1-1j & 0 \\ 2j & 4 & 2+2j & 0 \\ 1+1j & 2-2j & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because the fourth column of D is zero, the determinant should be zero.

$$\det(D) = 0$$
$$\det(E) = \det(7) = 7$$