Chapter 0, Mathematical Prerequisites: Problem Solutions

Problem 1

Recall that
\[ \sum_{i=0}^{k} z^i = \begin{cases} 
\frac{1-z^{k+1}}{1-z} & \text{for } z \neq 1 \\
N+1 & \text{for } z = 1.
\end{cases} \]

Proof for \( z \neq 0 \) (for \( z = 1 \) is trivial)

\[ s = 1 + z + z^2 + \ldots + z^N, \]
\[ -zs = -z - z^2 - \ldots - z^N - z^{N+1}. \]

Summing the above two equations gives
\[ (1 - z)s = 1 - z^{N+1} \Rightarrow s = \frac{1 - z^{N+1}}{1 - z}. \]

Similarly
\[ \sum_{k=N_1}^{N_2} z^k = z^{N_1} \sum_{k=0}^{N_2-N_1} z^k = \frac{z^{N_1} - z^{N_2} + 1}{1 - z}. \]

1. We have
\[ \sum_{n=1}^{N} s[n] = \sum_{n=1}^{N} 2^{-n} + j \sum_{n=1}^{N} 3^{-n} \]
\[ = \frac{1}{2} \cdot \frac{1 - 2^{-N}}{1 - 2^{-1}} + j \frac{1}{3} \cdot \frac{1 - 3^{-N}}{1 - 3^{-1}} = (1 - 2^{-N}) + j \frac{1}{2}(1 - 3^{-N}). \]

Now,
\[ \lim_{N \to \infty} 2^{-N} = \lim_{N \to \infty} 3^{-N} = 0. \]

Therefore,
\[ \sum_{n=1}^{\infty} s[n] = 1 + \frac{1}{2}j. \]

2. We can write
\[ \sum_{k=1}^{N} s[k] = \frac{j}{3} \cdot \frac{1 - (j/3)^N}{1 - j/3}. \]

Since \( \left| \frac{j}{3} \right| = \frac{1}{3} < 1 \), we have \( \lim_{N \to \infty} (j/3)^N = 0. \) Therefore,
\[ \sum_{k=1}^{\infty} s[k] = \frac{1}{3j - 1} = -\frac{1}{10} + j \cdot \frac{3}{10}. \]
3. From $z^* = z^{-1}$ with $z \in \mathbb{C}$, we have

$$zz^* = 1, \quad \forall z \neq 0.$$ 

Therefore, $|z|^2 = 1$ and, consequently, $|z| = 1$. It follows that all the $z$ such that $z^* = z^{-1}$ describe the unit circle.

4. Remark that $e^{2k\pi} = 1$, for all $k \in \mathbb{Z}$. Therefore, $z_k = e^{\frac{2k\pi}{3}}$ is such that $z_k^2 = 1$. Now $z_k$ is periodic of period 3, i.e. $z_k = z_{k+3l}$, for all $l \in \mathbb{Z}$. Therefore the (only) three different complex numbers are $z_0 = 1$, $z_1 = e^{\frac{2\pi}{3}}$ and $z_2 = e^{\frac{4\pi}{3}}$.

5. We have

$$\prod_{n=1}^{N} e^{j\frac{2\pi}{N}} = e^{j\pi \sum_{n=1}^{N} 2^{-n}} = e^{j\pi \frac{1-2^{-N}}{1-2^{-1}}}.$$ 

Since $\lim_{N \to \infty} 2^{-N} = 0$,

$$\prod_{n=1}^{\infty} e^{j\frac{2\pi}{N}} = e^{j\pi} = -1.$$ 

**Problem 2**

**[Geometric Series]**

(a)

- \[ S[n + 1] = S[n] + x[n + 1] \]
- \[ \tilde{S}[n] = rS[n] = r \cdot \sum_{k=0}^{n} a \cdot r^k = \sum_{k=1}^{n+1} a \cdot r^k = S[n+1] - a \]

\[
\begin{align*}
\begin{cases}
 rS[n] = S[n + 1] - a \\
 S[n + 1] = S[n] + a \cdot r^{n+1}
\end{cases}
\implies rS[n] = S[n] + a \cdot r^{n+1} - a
\implies S[n] = a \frac{1 - r^{n+1}}{1 - r}
\end{align*}
\]

(b)

\[ S = \sum_{n=0}^{\infty} x[n] = \lim_{n \to \infty} S[n] = \lim_{n \to \infty} a \frac{1 - r^{n+1}}{1 - r} = \frac{a}{1 - r} \]

(c)

\[
\sum_{k=n+1}^{m} x[k] = \sum_{k=0}^{m} x[k] - \sum_{k=0}^{n} x[k] = S[m] - S[n]
\]

\[ = a \frac{1 - r^{m+1}}{1 - r} - a \frac{1 - r^{n+1}}{1 - r} = a r^{n+1} - r^{m+1} \]

Based on part(a), we have

\[ \sum_{k=n+1}^{m} s[k] = \frac{a (m - n)}{1 - r} + a r^{n+2} \frac{1 - r^{m-n}}{1 - r} \]
To be more precise, if we set \( a' = \frac{ar^{n+2}}{1-r} \) we have,

\[
\sum_{k=n+1}^{m} \frac{ar^{k+1}}{1-r} = \sum_{k=0}^{m-n-1} a'r^k = a' \frac{1 - r^{m-n}}{1 - r} = ar^{n+2} \frac{1 - r^{m-n}}{(1-r)^2}
\]

(d)

\[
t[k] = \frac{1}{3^k} + \left(\frac{1}{2^j}\right)^k \implies \sum_{k=0}^{\infty} t[k] = \sum_{k=0}^{\infty} \frac{1}{3^k} + \sum_{k=0}^{\infty} \left(\frac{1}{2^j}\right)^k = \frac{1}{1 - \frac{1}{3}} + \frac{1}{1 - \frac{1}{2^j}} = \frac{23 - 4j}{10}
\]

(e) Define \( P = \prod_{n=1}^{\infty} e^{j\pi/2^n} \), and consider the fact that \( \ln(ab) = \ln(a) + \ln(b) \). Based on part (b) we have \( \sum_{n=1}^{\infty} \ln(e^{j\pi/2^n}) = \sum_{n=1}^{\infty} \frac{j\pi}{2^n} = j\pi \)

So we have

\[
P = e^{j\pi} = -1
\]

**Problem 3**

**[Complex Numbers]**

(a) **Remark**: If the summation over all coefficients in polynomial is zero then \( x = 1 \) is a root. If the summation over all coefficients of odd degree is equal to the summation of coefficient of even degree (e.g., the given polynomial in this problem) then \( x = -1 \) is a root. **Remark**: In every polynomial of order \( n \) in the general form \( P(x) = \sum_{k=0}^{n} a_k x^k \), \( a_n = 1 \) the summation of the roots is \(-a_{n-1}\). So, the summation of the roots is \(-2\) in this problem. We can also solve the equation as the following.

\[
x^3 + 2x^2 + 2x + 1 = (x + 1)(x^2 + x + 1) \implies x_1 = -1.
\]

In order to solve the remaining degree 2 polynomial, we can write

\[
\Delta = b^2 - 4ac = 1 - 4 = -3, \quad x_2, x_3 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 \pm \sqrt{3}}{2}
\]

In this case, the summation of the roots is \(-1 + \frac{-1 - \sqrt{3}}{2} + \frac{-1 + \sqrt{3}}{2} = -2\).

(b) We know that \( j = e^{j\pi/2} \)

\[
j^j = (e^{j\pi/2})^j = e^{-\pi}
\]

(c) Suppose \( z = re^{j\theta} \), \( r > 0 \)

\[
\arg(z) = |z| \\
\theta = r \\
z = \theta e^{j\theta} \quad 0 \leq \theta \leq 2\pi \\
z = \theta(\cos \theta + j \sin \theta)
\]

Fig. 1 illustrates all the point in the complex plane with this property.
(d) Characterize the set of complex numbers satisfying $z^* = z^{-1}$. Suppose $z = re^{j\theta}$, $r > 0$

$$z^* = z^{-1}$$

$$re^{-j\theta} = \frac{1}{r}e^{-j\theta}$$

$$r = \frac{1}{r}$$

$$r = 1 \quad \theta \in [0, 2\pi]$$

These numbers form a circle in the complex plane shown in Fig. 2.

![Figure 1: Problem 2(c): $z = \theta e^{j\theta}$](image1)

![Figure 2: Problem 2(d): $z = e^{j\theta}$](image2)

**Problem 4**

[LINEAR ALGEBRA]

(a) Compute the determinant for the following matrix.

$$A = \begin{bmatrix}
2 & 0 & -1 & 0 \\
1 & 0 & 2 & 1 \\
0 & 0 & 2 & 1 \\
-1 & -3 & 2 & 0
\end{bmatrix}$$

Since the second column of $A$ has only one non-zero element, it is easier to expand the determinant with respect to this column.

$$\det(A) = (-1)^{4+2}(-3) \det \begin{bmatrix} 2 & -1 & 0 \\
1 & 2 & 1 \\
0 & 2 & 1 
\end{bmatrix} =$$

By expanding with respect to the first column of the remaining matrix, we have

$$\det(A) = (-3)\left[(-1)^{1+1}(2) \det \begin{bmatrix} 2 & 2 \\
1 & 1 
\end{bmatrix} + (-1)^{2+1}(1) \det \begin{bmatrix} -1 & 0 \\
2 & 1 
\end{bmatrix}\right]$$

$$= (-3)[(+1)(2)(2 - 2) + (-1)(1)(-1 - 0)] = -3$$

(b) Consider the matrices

$$B = \begin{bmatrix}
 j & -1 & 4 \\
 0 & 2 - 3j & 1 \\
-1 & 2j & 0 \\
 3 & 0 & 4 - j
\end{bmatrix} \quad \quad C = \begin{bmatrix}
 0 & 0 & j & 1 \\
 1 - 5j & 1 & 4j & 2 + 2j \\
 1 & 3 - j & 0 & -7
\end{bmatrix}.$$

Which of the following operations are well-defined (Note that you do NOT have to compute)?

$C + B, C \cdot A^{-1}, B \cdot C, A - C, B + B^T, A + A^T, C^{-1} \cdot B^{-1}, C^* + B,$
Remark: Suppose matrix $A$ is $n \times m$ and matrix $B$ is $p \times q$. $A + B$ and $A - B$ are well-defined if and only if $p = n$ and $q = m$. $A \cdot B$ is well defined if and only if $m = p$. $A^{-1}$ exists if and only if $n = m$ and $\det(A) \neq 0$. $A^T$ and $A^*$ are $m \times n$ matrices.

So, the following matrices are well-defined. $C \cdot A^{-1}$, $B \cdot C$, $A + A^T$, $C^* + B$,

(c) Let $x = [1, 2j, 1 + j, 0]$. Compute $Ax^T$ and $xB$. $Ax^T = [1 - j, 3 + 2j, 2 + 2j, 1 - 4j]^T$

$xB = [-1, 3 + 6j, 4 + 2j]$

(d) Compute the determinant of $D = xx^T$ and $E = x^T x$.

$$E = [7]$$

$$D = \begin{bmatrix}
1 & 0 & -2j & 1 - 1j & 0 \\
2j & 4 & 2 + 2j & 0 \\
1 + 1j & 2 - 2j & 2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$  

Because the fourth column of $D$ is zero, the determinant should be zero.

$$\det(D) = 0$$

$$\det(E) = \det(7) = 7$$