Chapter 7, Linear Systems: Problem Solutions

Problem 1

[Filter Design: Parks-McClellan Algorithm]

(a) It is clear that $h[n] = h_e[n] + h_o[n]$ by the definition. We also have

$$h_e[-n] = \frac{1}{2}(h[-n] + h[n]) = \frac{1}{2}(h[n] + h[-n]) = h_e[n]$$

and

$$h_o[-n] = \frac{1}{2}(h[-n] - h[n]) = -\frac{1}{2}(-h[n] + h[n]) = -h_o[n]$$

which show $h_e[n]$ and $h_o[n]$ are even and odd sequences, respectively.

If $h[n]$ is causal, then $h[n] = 0$ for $n < 0$, and

$$h_e[n] = \frac{1}{2}(h[-n] + h[n]) = \frac{1}{2}(0 + h[n]) = \frac{1}{2}h[n] \qquad \text{for } n > 0$$

and

$$h_e[0] = \frac{1}{2}(h[0] + h[0]) = h[0],$$

which yields in

$$h[n] = 2h_e[n]u[n] - h_e[0]\delta[n]$$

which is true for general n.

For a real value sequence $h[n]$, the function $H(z)$ is a real function and is the same as its complex conjugate. Using the property $h[-n] \stackrel{DTFT}{\longleftrightarrow} H(e^{j(-\omega)})$, we have

$$h_e[n] = \frac{1}{2}(h[n] + h[-n]) \stackrel{DTFT}{\longleftrightarrow} \frac{1}{2}\left(H(e^{j\omega}) + H(e^{-j\omega})\right)$$

and

$$\frac{1}{2}\left(H(e^{j\omega}) + H(e^{-j\omega})\right) = \frac{1}{2}\left(H(e^{j\omega}) + (H(e^{-j\omega}))^*\right) = \Re\{H(e^{j\omega})\}$$

(b) Starting from the set of equations

$$W(e^{j\omega_n})(H_{dr}(e^{j\omega_n}) - P(e^{j\omega_n})) = (-1)^n\delta_2 \text{ for } n = 0, 1, 2, ..., L+1,$$

we have

$$H_{dr}(e^{j\omega_n}) = \frac{(-1)^n\delta_2}{W(e^{j\omega_n})} + P(e^{j\omega_n}) = \frac{(-1)^n\delta_2}{W(e^{j\omega_n})} + \sum_{k=0}^{L} a\,[k]\cos(\omega_n k),$$

which can be easily rewritten in matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_0) & \cos(2\omega_0) & ... & \cos(L\omega_0) & \frac{1}{W(\omega_0)} \\ 1 & \cos(\omega_1) & \cos(2\omega_1) & ... & \cos(L\omega_1) & \frac{-1}{W(\omega_1)} \\ ... & & & & & \\ 1 & \cos(\omega_{L+1}) & \cos(2\omega_{L+1}) & ... & \cos(L\omega_{L+1}) & \frac{(-1)^{L+1}}{W(\omega_{L+1})} \end{bmatrix} \begin{bmatrix} a\,[0] \\ a\,[1] \\ ... \\ a\,[L] \\ \delta_2 \end{bmatrix} = \begin{bmatrix} H_{dr}(e^{j\omega_0}) \\ H_{dr}(e^{j\omega_1}) \\ ... \\ H_{dr}(e^{j\omega_{L+1}}) \end{bmatrix}$$

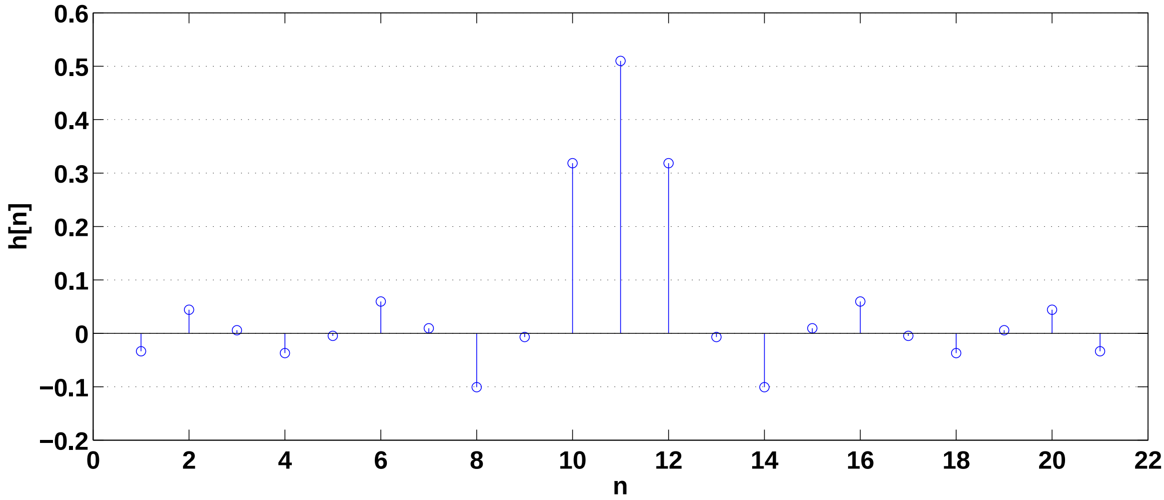

Figure 1: Impulse response.

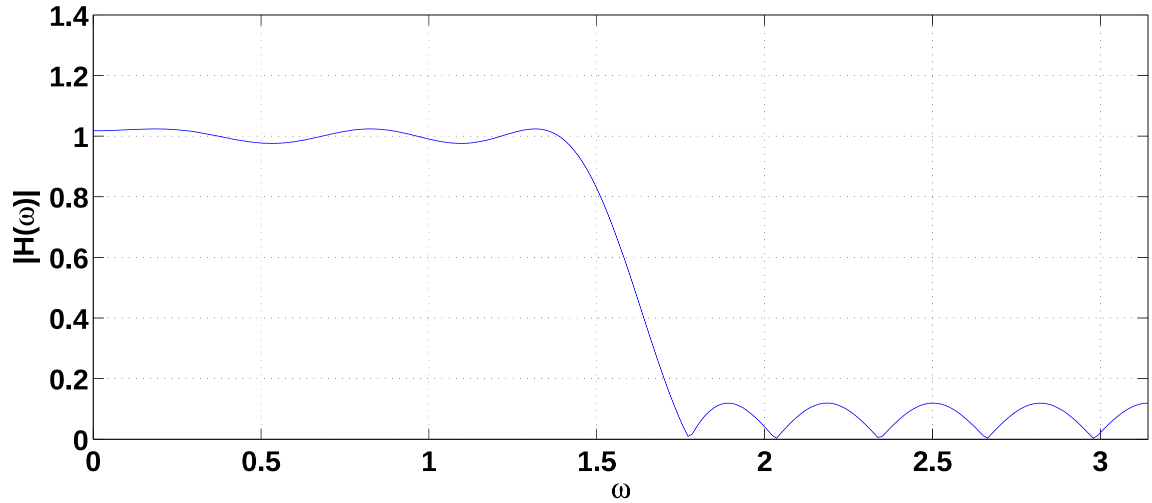

Figure 2: Frequency response..

where the first $L+1$ columns of the matrix are corresponding to the term $\sum_{k=0}^{L} a\left[k\right]\cos(\omega_n k)$ and the last column is the contribution of the term $\frac{(-1)^n \delta_2}{W(e^{j\omega_n})}$.

c You can use the follwing MATLAB code to plot the impulse response and frequency response of the desired filter.

```
 [h,err]=firpm(20,[0 0.45 0.55 1],[1 1 0 0],[5 1]);
subplot(2,1,1);
stem(h);
H=fft(h,500);
subplot(2,1,2);
plot([0:499]*2*pi/500,abs(H));
axis([0 pi 0 1.2]);
xlabel('
omega');
ylabel('|H(
omega)|');
```

Problem 2

1. $D\{\alpha x[n]\} = \alpha x[n-1] = \alpha D\{x[n]\}$
   $D\{x[n] + y[n]\} = x[n-1] + y[n-1] = D\{x[n]\} + D\{y[n]\}$.

2. Δ is a *linear combination* of the original signal with the linear operator D, therefore it is also linear.

3. $S\{\alpha x[n]\} = \alpha^2 x^2[n-1] = \alpha^2 S\{x[n]\} \neq \alpha S\{x[n]\}$.

4. $$\mathbf{\Delta} = \mathbf{I} - \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

5. The matrix realizes an integration operation over a vector in $\mathbb{C}^4$.

Problem 3

1. $y[n] = x[-n]$

   Linear: $\mathcal{H}\{ax_1[n] + bx_2[n]\} = ax_1[-n] + bx_2[-n] = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}$. Therefore, $\mathcal{H}$ is linear.

   Time Invariant: $\mathcal{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0]$. Therefore, $\mathcal{H}$ is NOT time invariant.

   Stable: If $|x[n]| \leq M$, then $|\mathcal{H}\{x[n]\}| \leq M$. Therefore, $\mathcal{H}$ is BIBO stable.

   Causal: $\mathcal{H}$ is not causal.

   Impulse response: $\mathcal{H}$ is not LTI, therefore $h[n]$ does not characterize the system.

2. $y[n] = e^{-j\omega n} x[n]$

   Linear: $\mathcal{H}\{ax_1[n] + bx_2[n]\} = e^{-j\omega n}(ax_1[n] + bx_2[n]) = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}$. Therefore, $\mathcal{H}$ is linear.

   Time Invariant: $\mathcal{H}\{x[n-n_0]\} = e^{-j\omega n} x[n-n_0] = e^{j\omega n_0} y[n-n_0]$. Therefore, $\mathcal{H}$ is not time invariant (only for $\omega = 0$).

   Stable: If $|x[n]| \leq M$, then $|\mathcal{H}\{x[n]\}| = |x[n]| \leq M$. Therefore, $\mathcal{H}$ is BIBO stable.

   Causal: $\mathcal{H}$ is causal.

   Impulse response: $\mathcal{H}$ is not LTI, therefore $h[n]$ does not characterize the system.

3. $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$

   Linear: $\mathcal{H}\{ax_1[n] + bx_2[n]\} = \sum_{k=n-n_0}^{n+n_0} (ax_1[k] + bx_2[k]) = a\mathcal{H}\{x_1[n]\} + b\mathcal{H}\{x_2[n]\}$. Therefore, $\mathcal{H}$ is linear.

   Time Invariant: $\mathcal{H}\{x[n-n_0]\} = \sum_{k=n-n_0}^{n+n_0} x[k-n_0] = \sum_{k=n-2n_0}^{n} x[k] = y[n-n_0]$. Therefore, $\mathcal{H}$ is time invariant.

   Stable: If $|x[n]| \leq M$, then $\mathcal{H}\{x[n]\} \leq |2n_0 + 1|M$. Therefore, $\mathcal{H}$ is BIBO stable.

   Causal: $\mathcal{H}$ is not causal.

   Impulse response: If $x[n] = \delta[n]$, $y[n] = h[n]$:

   $$h[n] = \begin{cases} 1 & \text{if } |n| \leq |n_0|, \\ 0 & \text{otherwise.} \end{cases}$$

4. $y[n] = ny[n-1] + x[n]$, such that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.

   Since $\mathcal{H}$ is recursive, we can not use the same technique as before. Note that all inputs $x[n]$ can be expressed as a linear combination of delayed impulses: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$. Therefore, to show that $\mathcal{H}$ is linear or time invariant, we can restrict the input to delayed impulses.

   If $x[n] = \delta[n]$, we can obtain $y[n]$ by recursion:

$$h[n] = y[n] = n!u[n].$$

   If $x[n] = a\delta[n] + b\delta[n]$:

$$y[n] = (a+b)n!u[n].$$

   Therefore, $\mathcal{H}$ is linear.

   To check if $\mathcal{H}$ is time invariant, consider $x[n] = \delta[n-1]$. It is easy to check that $\mathcal{H}\{\delta[n-1]\}! = h[n-1]$.

   Stable: The system is non stable.

   Causal: $\mathcal{H}$ is causal.

   Impulse response: $\mathcal{H}$ is not LTI, therefore $h[n]$ does not characterize the system.

Problem 4

1. Consider the sequence $x[n] = \delta[n-1]$; we should have $\mathcal{R}\{x[n]\}[n] = \mathcal{R}\{\delta[n]\}[n-1]$ but instead it is

$$\begin{aligned} \mathcal{R}\{x[n]\}[n] &= x[-n] = \delta[-(n+1)] = \delta[n+1] \\ \mathcal{R}\{\delta[n]\}[n-1] &= \delta[n-1] \end{aligned}$$

2. First of all recall that the DTFT of $x[-n]$ is $X(e^{-j\omega})$; if $x[n]$ is real, we also have $X(e^{j\omega}) = X^*(e^{-j\omega})$. In the frequency domain we therefore have:

   (a) $S(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

   (b) $R(e^{j\omega}) = S(e^{-j\omega}) = H^*(e^{j\omega})X(e^{-j\omega})$ since $h[n]$ is real.

   (c) $W(e^{j\omega}) = H(e^{j\omega})R(e^{j\omega}) = |H(e^{j\omega})|^2 X(e^{-j\omega})$

   (d) $Y(e^{j\omega}) = W(e^{-j\omega}) = |H(e^{j\omega})|^2 X(e^{j\omega})$

   Therefore the chain of transformations defines an LTI filter $\mathcal{G}$ with frequency response $G(e^{j\omega}) = |H(e^{j\omega})|^2$. The corresponding impulse response is simply

$$g[n] = h[n] * h[-n]$$

   What is interesting to note here is that, even though $\mathcal{R}$ is not time invariant, we can combine time variant operators into an overall time-invariant transformation.

3. $G(e^{j\omega})$ is a real function, therefore its phase is zero.