EPFL Winter Semester 2007/2008
Handout # 15, Wednesday, October 24th, 2007

Chapter 4, Signals and Hilbert Spaces: Problem Solutions

Problem 1

1. Recall that the set of N non-zero orthogonal vectors in an N-dimensional subspace is a basis for the subspace. Therefore, we need to prove the orthogonality across the vectors $\{\mathbf{w}^{(k)}\}_{k=0,\dots,N-1}$. Let us compute the inner product, that is:

$$
\begin{aligned}
\langle \mathbf{w}^{(k)}, \mathbf{w}^{(h)} \rangle &= \sum_{n=0}^{N-1} \mathbf{w}^{(k)} \mathbf{w}^{*(h)} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} e^{j\frac{2\pi}{N}nh} \\
&= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(k-h)} = \begin{cases} N & \text{if } k = h \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

   Since the inner product of the vectors is equal to zero, we conclude that they are orthogonal. However, they do not have a unit norm and therefore are not the orthonormal vectors.

2. In order to obtain the *orthonormal basis* we normalize the vectors with the factor $1/\sqrt{N}$, having:

$$
\begin{aligned}
\langle \mathbf{w}^{(k)}_{norm}, \mathbf{w}^{(h)}_{norm} \rangle &= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}nk} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nh} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(k-h)} = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$