Chapter 6, Fourier Analysis - Practice: Problem Solutions

Problem 1

[Fast Fourier Transform]

(a) In order to computing $X(k)$, a summation over N terms should be computed which needs $N-1$ addition. On the hand, for computing each term in the summation, we need N multiplication. Since all these process should be repeated for any $X(k)$ for $k = 0, \ldots, N-1$, the total number of additions and multiplications are $N(N-1)$ and N^2, respectively.

(b)

$$\begin{aligned}
X(p,q) &= X(Mp+q) = \sum_{n=0}^{N-1} x(n) W_N^{(Mp+q)n} \\
&\overset{(*)}{=} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x(l+mL) W_N^{(Mp+q)(l+mL)} \\
&= \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x(l+mL) W_N^{pmML+lpM+mqL+lq} \\
&\overset{(**)}{=} \sum_{l=0}^{L-1} \left[\sum_{m=0}^{M-1} x(l+mL) W_N^{mqL} \right] W_N^{lq} W_N^{lpM}
\end{aligned}$$

where n is replaced by $(l+mL)$ in $(*)$, and $(**)$ follows from the fact that $W_N^{pmML} = (W_N^N)^{pm} = 1$. Note that $W_N^L = (e^{j\frac{2\pi}{ML}})^L = e^{j\frac{2\pi}{M}} = W_M$ and $W_N^M = (e^{j\frac{2\pi}{ML}})^M = e^{j\frac{2\pi}{L}} = W_L$. Therefore we have

$$\begin{aligned}
X(p,q) &= \sum_{l=0}^{L-1} \left[\sum_{m=0}^{M-1} x(l+mL) W_M^{mq} \right] W_N^{lq} W_L^{lp} \\
&= \sum_{l=0}^{L-1} \left\{ W_N^{lq} \left[\sum_{m=0}^{M-1} x(l,m) W_M^{mq} \right] \right\} W_L^{lp}
\end{aligned}$$

(c) In order to compute $X(p,q)$ for $p = 0, \ldots, L-1$ and $q = 0, \ldots, M-1$, one has to compute $F(l,q)$, $G(l,q)$, and the final expression for all combination of l and q.

- $F(l,q)$: The summation is over M terms, so we need M multiplication and $M-1$ additions for each pair (l,q). Thus totally $ML(M-1)$ additions and MLM multiplication is required.
- $G(l,q)$: There is only one multiplication for fixed (l,q). Therefore we need only ML multiplications.

- $X(p,q)$: Once $G(l,q)$ be available, $X(p,q)$ can be computed by $L-1$ addition and L multiplications for fixed (p,q), which results in $LM(L-1)$ additions and LML multiplications in total.

Finally the total numbers of additions and multiplications required in this process are

$$ML(M-1) + 0 + ML(L-1) = N(M+L-2)$$

and

$$MLM + ML + MLL = N(M+1+L),$$

respectively.

For the given numbers $N = 1000$, $L = 2$, and $M = 500$, the total number of additions and multiplications are given in the following table. It is clear that both the numbers are decreased by factor of two (approximately).

	addition	multiplication
part (a)	999000	1000000
part (c)	500000	503000