Chapter 9, Filter Design: Problem Solutions

Problem 1

1. First of all note that $1 - 2^{-k} = (2^k - 1)/2^k$. With this we find that

$$\begin{aligned} z_1 &= e^{j\frac{1}{2}\pi} \\ z_2 &= e^{j\frac{3}{4}\pi} \\ z_3 &= e^{j\frac{7}{8}\pi} \\ z_4 &= e^{j\frac{15}{16}\pi} \end{aligned}$$

   which are simply four points in the second quadrant on the unit circle.

2. Each couple of complex-conjugate zeros contributes a factor of the form $(1-2z^{-1}\cos\theta + z^{-2})$ to the transfer function, where θ is the angle of the complex zero. We have in the end:

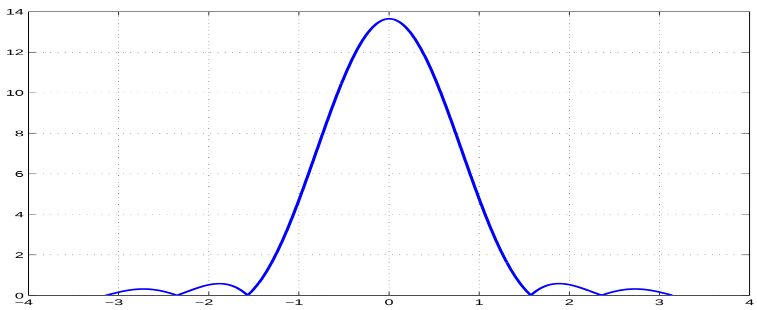
$$H(z) = (1 + z^{-1})(1 + z^{-2})(1 - 2\cos(\frac{3}{4}\pi)\, z^{-1} + z^{-2})$$

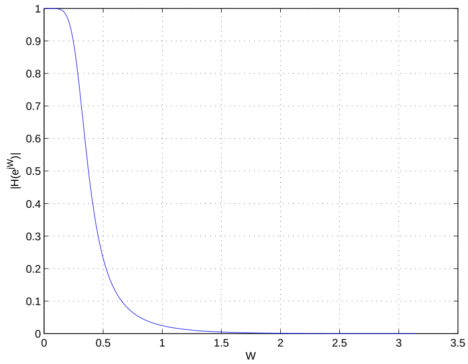
3. $H(z)$ is a 5^{th} degree polynomial in z^{-1} and therefore it has at most 6 nonzero coefficients. The impulse response will have 6 nonzero taps.

4. You don't even need Matlab to do this. First of all, the impulse response is real and therefore the magnitude of $H(e^{j\omega})$ is symmetric. Consider now the values of the frequency response at zero and π; these are computed from the z-transform for $z = 1$ and $z = -1$ respectively; we have:

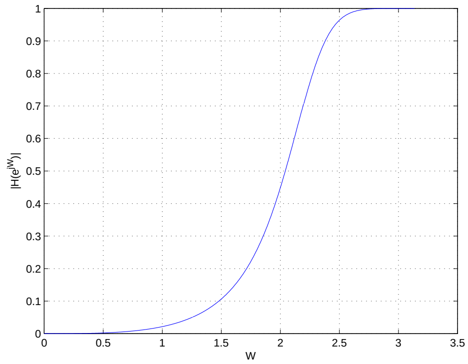
$$\begin{aligned} H(e^{j0}) &= H(1) = 2 \cdot 2 \cdot 2(1 - \cos(\frac{3}{4}\pi)) \approx 13.6 \\ H(e^{j\pi}) &= H(-1) = 0 \end{aligned}$$

   Next, you need to consider that $H(z)$ is zero on the unit circle at z_1 and z_2, i.e. at $\omega = \pi/2$ and $\omega = 3\pi/4$. Now you can plot the magnitude:

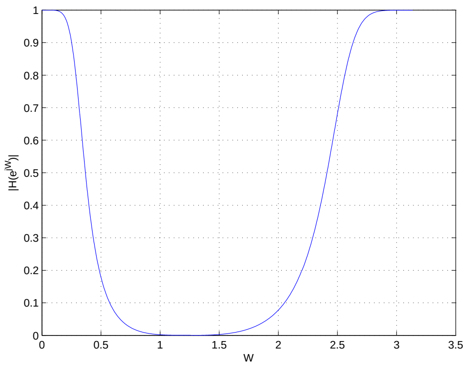




(a) Diagram 1



(b) Diagram 2



(c) Diagram 3

Figure 1: Zeros and Poles Diagrams

5. First of all, is the filter linear phase? You can compute the coefficient of the transfer function and verify that $h[n] = 1, 2.4142, 3.4142, 3.4142, 2.4142, 1$ for $n = 0, \ldots, 5$. In a simpler way, you can simply notice that $H(z^{-1}) = z^5 H(z)$ and therefore the filter is linear phase, symmetric. The filter has an even number of taps and therefore it is Type II.

   Because of the zero in π and the large value in zero, the filter is lowpass. However, it is not equiripple since the magnitude at the peak of the first sidelobe in the stopband is higher than the peak of the second sidelobe.

   The filter is clearly not a good filter: the transition band is very large, it is not flat in the passband and the magnitude is rather large in the stopband.

Problem 2

To obtain the frequency response of a filter, we analyze the z-transform in the unit circle, that is, in $z = e^{j\omega}$. Figure 1 shows the exact magnitude of each frequency response:

1. The first filter is a low-pass filter. Note that there are three poles located in low frequency (near $\omega = 0$), while there is a zero located in high frequency ($\omega = \pi$).

2. The second filter is just the opposite. The zero is located in low frequency, while the influence of the three poles is maximum in high frequency ($\omega = \pi$). Therefore, it is a high-pass filter.

3. In the third system, there are poles which affect low and high frequency and two zeros close to $w = \pi/2$. Therefore, this system is a stop-band filter.

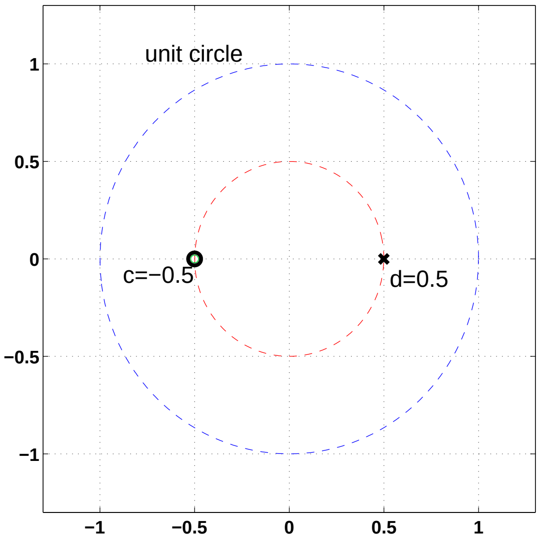


Fig. 2: The zero-pole plot of $H(z)$ for $\phi = 0$.

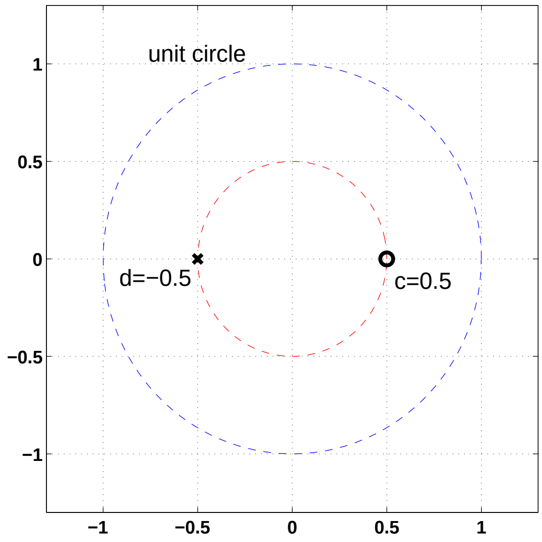


Fig. 3: The zero-pole plot of $H(z)$ for $\phi = \pi$.

Problem 3

(a) Clearly, $c = \frac{1}{2}e^{j(\phi+\pi)}$ and $d = \frac{1}{2}e^{j\phi}$ are the zero and the pole of $H(z)$, respectively (see Figures 2 and 3).

(b) Since the system is LTI, the output of the system can be written as

$$y[n] = h[n] * x[n]$$

in terms of the input and impulse response. Therefore, we have

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

and by taking logarithm of the magnitude,

$$20\log_{10}\left|Y\left(e^{j\omega}\right)\right| = 20\log_{10}\left|H\left(e^{j\omega}\right)\right| + 20\log_{10}\left|X\left(e^{j\omega}\right)\right|$$

(c) Using the property $|A|^2 = AA^*$, we have

$$\begin{aligned}
20\log_{10}|1 - re^{j\theta}| &= 10\log_{10}(1 - re^{j\theta})(1 - re^{j\theta})^* \\
&= 10\log_{10}(1 - re^{j\theta})(1 - re^{-j\theta}) \\
&= 10\log_{10}\left(1 - r\cos(-\theta) - r\sin(-\theta) - r\cos(\theta) - r\sin(\theta) + r^2\right) \\
&= 10\log_{10}(1 + r^2 - 2r\cos(\theta))
\end{aligned}$$

(d)

$$\begin{aligned}
20\log_{10}\left|H\left(e^{j\omega}\right)\right| &= 20\log_{10}\left|\frac{1-ce^{-j\omega}}{1-de^{-j\omega}}\right| \\
&= 20\log_{10}\left|1-ce^{-j\omega}\right| - 20\log_{10}\left|1-de^{-j\omega}\right| \\
&= 20\log_{10}\left|1-\frac{1}{2}e^{j(\phi+\pi)}e^{-j\omega}\right| - 20\log_{10}\left|1-\frac{1}{2}e^{j\phi}e^{-j\omega}\right| \\
&= 10\log_{10}(1+\frac{1}{4}-2\frac{1}{2}\cos(\phi+\pi-\omega)) - 10\log_{10}(1+\frac{1}{4}-2\frac{1}{2}\cos(\phi-\omega)) \\
&= 10\log_{10}(\frac{5}{4}+\cos(\phi-\omega)) - 10\log_{10}(\frac{5}{4}-\cos(\phi-\omega)) \\
&= 10\log_{10}\left(\frac{\frac{5}{4}+\cos(\phi-\omega)}{\frac{5}{4}-\cos(\phi-\omega)}\right)
\end{aligned}$$

For $\omega=\phi$, we have

$$20\log_{10}\left|H\left(e^{j\omega}\right)\right|\Big|_{\omega=\pi} = 10\log_{10}\left(\frac{\frac{5}{4}+\cos(0)}{\frac{5}{4}-\cos(0)}\right) = 10\log_{10}(9) = 9.542$$

which is a large magnitude and shows that system amplify the input at this frequency.

For $\omega=\phi+\pi$,

$$[20\log_{10}\left|H\left(e^{j\omega}\right)\right|\Big|_{\omega=\phi+\pi} = 10\log_{10}\left(\frac{\frac{5}{4}+\cos(\pi)}{\frac{5}{4}-\cos(\pi)}\right) = 10\log_{10}(\frac{1}{9}) = -9.542$$

which shows that the system is behaves as a filter and pass only a small amount of the energy of the input at this frequency.

(e) Here is the MATLAB code

```
phi=0;
c=0.5*exp(i*(pi+phi));
d=0.5*exp(i*phi);
w=0:0.01:2*pi;
z=exp(i*w);
H=(1-c*z.^(-1))./(1-d*z.^(-1));
plot(w,20*log10(abs(H)));
```

which results in the plot given in Fig. 4, and shows that the filter is low-pass.

(f) The same code (except `phi=pi;`) can be used. The plot is illustrated in Fig. 5, and shows that the filter is high-pass.

(g)

$$H(z) = \frac{1-cz^{-1}}{1-dz^{-1}} \overset{(*)}{=} (1-cz^{-1})\cdot\sum_{n=0}^{\infty}\left(dz^{-1}\right)^n$$

where $(*)$ follows from the region of convergence of $H(z)$. Therefore,

$$h[n] = \begin{cases} 0 & \text{if } n<0 \\ 1 & \text{if } n=0 \\ d^{n-1}(d-c) & \text{if } n\geq 1 \end{cases}$$

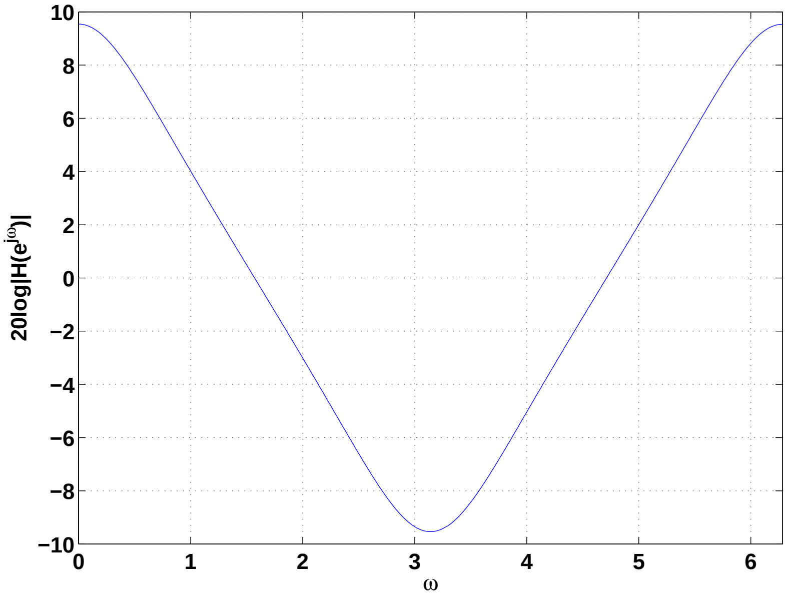


Figure 4: $20 \log_{10} |H(e^{j\omega})|$ for $\phi = 0$.

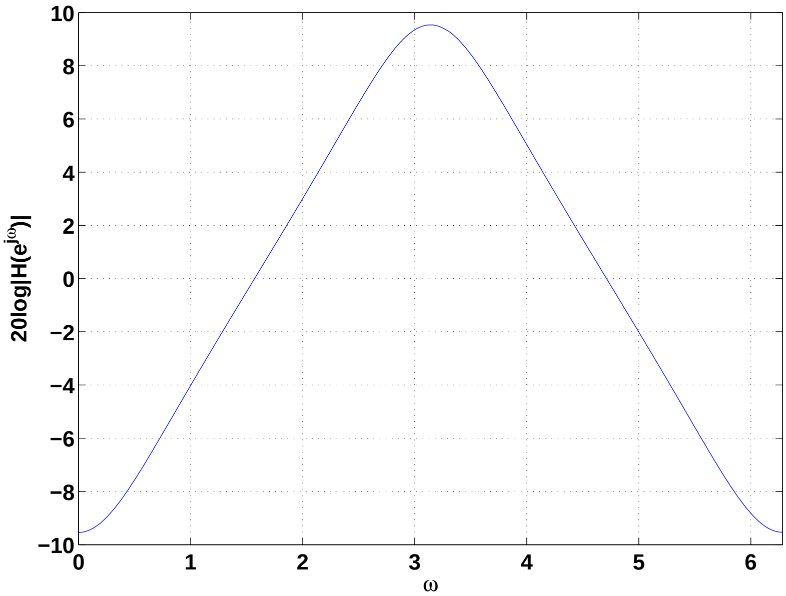


Figure 5: $20 \log_{10} |H(e^{j\omega})|$ for $\phi = \pi$.

Problem 4

1. First, read in the signal, compute its DFT and plot the amplitude of the spectrum:

```
>> [s,fs]=wavread('santa_corrupt.wav')
>> S=fft(s);
>> set(axes('FontSize',32))
>> plot(2*pi*[0:20093]/20094,abs(S))
>> xlabel('\omega')
>> ylabel('S(e^{j\omega})')
```

The resulting plot is shown in Fig. 6. It can clearly be seen that noise at a frequency

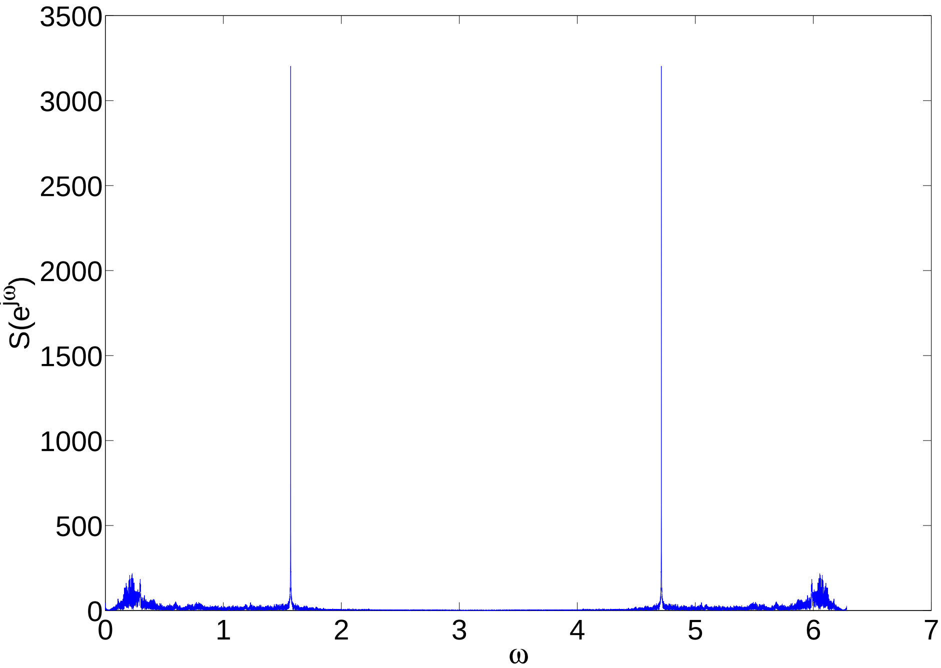


Figure 6: Spectrum of the santa_corrupt sequence

slightly above $\frac{\pi}{2}$ corrupts the signal.

2. We will use the Parks-Mclellan algorithm to design a 21-tap lowpass filter with cutoff frequency around 0.4π. Then, we filter the sequence to denoise it.

```
>> [h,err]=firpm(20,[0 0.35 0.45 1],[1 1 0 0],[1 20]);
>> x=filter(h',1,s');
>> soundsc(x,fs)
```

We allow larger ripples in the passband than in the stopband to make sure that we strongly attenuate the noise. In Fig. 7, we show the spectrum of the denoised signal, while in Fig. 8, we show the time and amplitude response of the filter we designed.

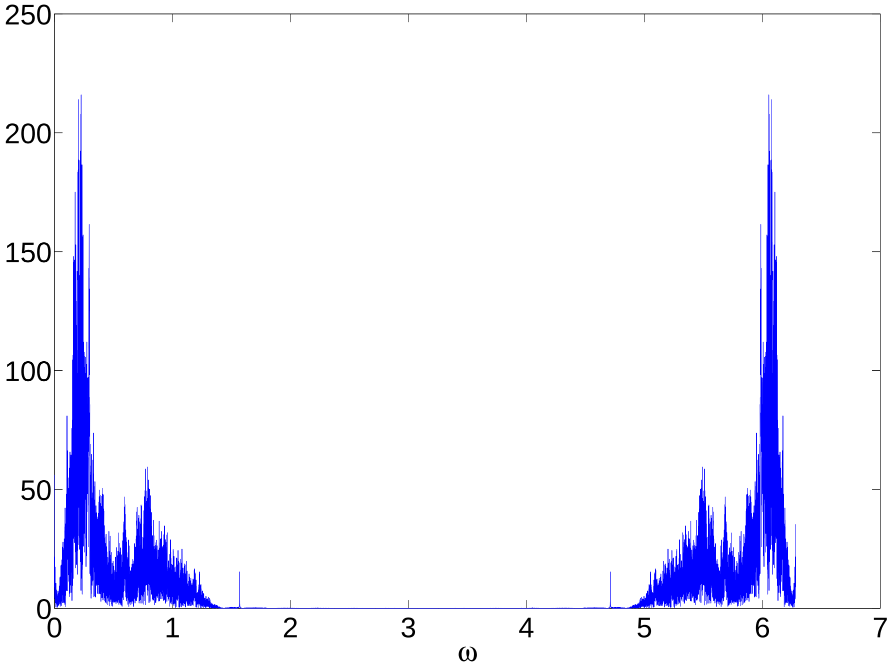
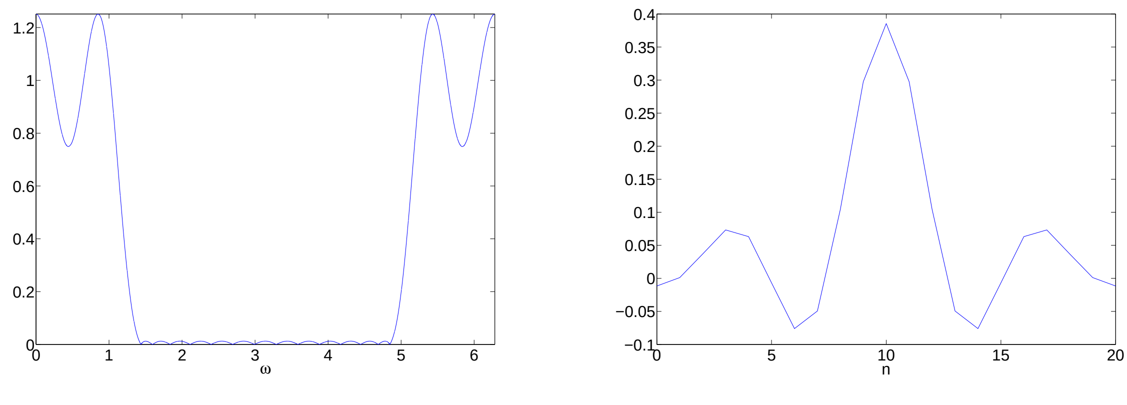


Figure 7: Spectrum of the denoised sequence



(a) Amplitude response

(b) Time response

Figure 8: Amplitude and responses of the filter we designed using the Parks-McLellan algorithm