Chapter 5, The DTFT (Discrete-Time Fourier Transform): Problem Solutions

Problem 1

1. The inner product in $l_2(\mathbb{Z})$ is defined as

$$\langle x[n], y[n] \rangle = \sum_{n} x^*[n] y[n],$$

and in $L_2([-\pi, \pi])$ as

$$\langle X(e^{jw}), Y(e^{jw}) \rangle = \int_{-\pi}^{\pi} X(e^{jw})^* Y(e^{jw}) dw.$$ 

Thus,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})^* Y(e^{jw}) dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sum_n x[n] e^{-jwn})^* (\sum_m y[m] e^{-jwm}) dw$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sum_n x[n] e^{-jwn})^* y[n] e^{-jwn} dw$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sum_m y[m] e^{-jwn})^* y[n] e^{-jwn} dw$$

$$= \frac{1}{2\pi} \sum_n \sum_m x^*[n] y[m] \int_{-\pi}^{\pi} e^{jw(n-m)} dw$$

$$= \sum_n x^*[n] y[n],$$

where (1) follows from the properties of the complex conjugate, (2) follows from swapping the integral and the sums and (3) from the fact that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jw(n-m)} dw = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n.
\end{cases}$$

2. If $x[n] = y[n]$, then $\langle x[n], x[n] \rangle$ corresponds to the energy of the signal in the time domain and $\langle X(e^{jw}), X(e^{jw}) \rangle$ to the energy of the signal in the frequency domain. In this case, the Plancherel-Parseval equality illustrates an energy conservation property from the time domain to the frequency domain. This property is known as the Parseval theorem.

Problem 2

[DFT and DTFT]
Figure 1: Problem 5(a).

(a) 
\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=3}^{9} e^{-j\omega n} - \sum_{n=10}^{14} e^{-j\omega n} = e^{-j\omega 3} \frac{1 - e^{-j\omega 7}}{1 - e^{-j\omega}} - e^{-j\omega 10} \frac{1 - e^{-j\omega 5}}{1 - e^{-j\omega}} = e^{-j\omega 3} \frac{2 e^{-j\omega 10} + e^{-j\omega 15}}{1 - e^{-j\omega}},
\]

for \( \omega \neq 0 \). For \( \omega = 0 \) we have \( X(e^{j0}) = \sum_{n=3}^{9} 1 - \sum_{n=10}^{14} 1 = 2 \).

(b) See Figure 1 that was obtained by

\[
\begin{align*}
&\text{>> w = linspace(0,2*pi,1e3+1);} \\
&\text{>> w = w(1:end-1); \% we don’t want 2*pi itself} \\
&\text{>> X = (exp(-i*w*3)-2*exp(-i*w*10)+exp(-i*w*15))./(1-exp(-i*w));} \\
&\text{Warning: Divide by zero.} \\
&\text{>> X(1) = 2;} \\
&\text{>> plot(w,abs(X))} \\
&\text{>> xlim([0 2*pi])} \\
&\text{>> xlabel('\omega')} \\
&\text{>> ylabel('|X(e^{j\omega})|')} \\
\end{align*}
\]

(c)-(d) See Figure 2, where we give the result for \( N = 100 \) only:
Figure 2: Problem 6(c)-(d).

We see that the DFT sequence corresponds exactly to points on the DTFT curve.

Problem 3

1. The discrete-time sequence $x[n]$ can be written as the convolution of $x_1[n]$ and $x_2[n]$ defined as

$$x_1[n] = x_2[n] = \begin{cases} 1 & -(M-1)/2 \leq n \leq (M-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

In fact,

$$x_1[n] * x_2[n] = \sum_k x_1[k] x_2[n-k]$$

$$\overset{(1)}{=} \sum_k x_1[k] x_1[k-n]$$

$$\overset{(2)}{=} x[n]$$
Figure 3: The discrete-time sequence $x[n]$ for $M = 11$.

where (1) follows from the fact that $x_1[n] = x_2[n]$ and from the symmetry of $x_1[n]$ and (2) noticing that the sum corresponds to the size of the overlapping area between $x_1[k]$ and its $n$-shifted version $x_1[k-n]$. When $|n| \geq M$ the two sequences do not overlap whereas the size of the overlapping area reaches its maximum $M$ when $n = 0$.

Using Matlab, we can easily verify the above result for $M = 11$ using the following code:

```matlab
>> M = 11;
>> x1 = ones(1,M);
>> x2 = x1;
>> x = conv(x1,x2);
>> stem([-M+1:M-1], x);
```

The result is shown in Figure 3.

2. Note that $x_1[n] = u[n + (M - 1)/2] - u[n - (M + 1)/2]$. We can thus compute its DTFT as

$$X_1(e^{j\omega}) \overset{(1)}{=} \left( \frac{1}{1 - e^{-j\omega}} + \frac{1}{2} \delta(\omega) \right) \left( e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2} \right)$$

$$\overset{(2)}{=} \frac{e^{j\omega(M-1)/2} - e^{-j\omega(M+1)/2}}{1 - e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega M/2} - e^{-j\omega M/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}$$

$$= \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

where (1) follows from the DTFT of $u[n]$ and (2) from the fact that

$$e^{j\omega(M-1)/2} \delta(w) = e^{-j\omega(M+1)/2} \delta(w) = \delta(w).$$

Using the convolution theorem, we can write

$$X(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

$$= X_1(e^{j\omega})X_1(e^{j\omega})$$

$$= \left( \frac{\sin(\omega M/2)}{\sin(\omega/2)} \right)^2.$$
Problem 4

1. \( \mathcal{H}\{\delta[n]\} = \delta[n] \); but \( \mathcal{H}\{a\delta[n]\} = a^2\delta[n] \neq a\mathcal{H}\{\delta[n]\} \).

2. Let \( y[n] = \mathcal{H}\{x[n]\} \); let \( w[n] = x[n - n_0] \); \( \mathcal{H}\{w[n]\} = w^2[n] = x^2[n - n_0] = y[n - n_0] \).
   QED.

3. First of all, \( y[n] = \cos^2(\omega_0 n) = (1 + \cos(2\omega_0 n))/2 \) from the well-known trigonometric identity. So \( y[n] \) contains a sinusoid at \textit{double} the original frequency (but be careful: double in the \(2\pi\)-periodic sense: if \( \omega_0 \) is larger than \( \pi/2 \), then \( 2\omega_0 \) will wrap around the \([-\pi, \pi]\) interval).

   If \( \omega_0 = 3\pi/8 \), then \( y[n] = (1 + \cos((3\pi/4)n))/2 \); since \( \mathcal{G} \) is a highpass with cutoff frequency \( \pi/2 \), it will kill the frequency components below \( \pi/2 \) and therefore it will kill the constant. The only component that passes through is the cosine at \( 3\pi/4 \). The final output is therefore \( v[n] = \frac{1}{2} \cos((3\pi/4)n) \).

4. If \( \omega_0 = 7\pi/8 \), then \( 2\omega_0 = 7\pi/4 > \pi \). We can therefore bring back the frequency into the \([-\pi, \pi]\) interval. We have that \( 7\pi/4 = 2\pi - \pi/4 \) and therefore \( \cos((7\pi/4)n) = \cos((2\pi - \pi/4)n) = \cos((\pi/4)n) \). So in the end \( y[n] = (1 + \cos((\pi/4)n))/2 \). Now the frequency of the cosine is below \( \pi/2 \) and therefore \( v[n] = 1 + \cos((\pi/4)n) \). Note that, as for most nonlinear systems, the frequency of the input sinusoid is different from the frequency of the output sinusoids: sinusoids are no longer eigenfunctions!