Random matrix theory: lecture 9

Asymptotic analysis: first approach

Observation: the capacity of a $n \times n$ MIMO channel whose channel matrix H has i.i.d. $N_{\mathbb{C}}(0,1)$ entries is given by

$$C = \mathbb{E}\left(\log \det\left(I + \frac{P}{n} HH^*\right)\right)$$

Notice that, for each j fixed, $\left(\frac{1}{n} HH^*\right)_{jj} = \frac{1}{n}\sum_{\ell=1}^{n} |h_{j\ell}|^2 \xrightarrow[n\to\infty]{} \mathbb{E}(|h_{jj}|^2) = 1$

by the law of large numbers

Similarly $\left(\frac{1}{n} HH^*\right)_{jk} = \frac{1}{n}\sum_{\ell=1}^{n} h_{j\ell}\overline{h_{k\ell}} \xrightarrow[n\to\infty]{} \mathbb{E}(h_{j1}\overline{h_{k1}}) = 0 \quad \forall j \neq k$

Therefore, $\frac{1}{n} HH^* \underset{n\to\infty}{\sim} I$, so

$$C \underset{n\to\infty}{\sim} \mathbb{E}\left(\log\det(I + PI)\right) = n\log(1+P) \quad ???$$

Why would we need random matrix analysis, then?

The <u>problem</u> is that the determinant (and likewise, the eigenvalues) of a matrix is not a continuous function of the entries, as the size of the matrix goes to infinity. Therefore, even though $\frac{1}{n} HH^*$ converges entrywise to I as $n\to\infty$, $\det(I + \frac{P}{n} HH^*)$ diverges from $\det(I + PI)$.

Here is another simple (counter-) example: let $A = (1-\frac{1}{n}) I$ of size $n \times n$

Then $A \underset{n\to\infty}{\sim} I$, but $\det A = (1-\frac{1}{n})^n \underset{n\to\infty}{\sim} \frac{1}{e} \neq \det I = 1$.

General formulation of the problem

Let $(A^{(n)})_{n\geq 1}$ be a sequence of random matrices of increasing size ($A^{(n)}$ is of size $n\times n$), $\lambda_1^{(n)} \dots \lambda_n^{(n)}$ denote the (ordered or un-ordered) eigenvalues of $A^{(n)}$ and let $\lambda^{(n)}$ be one eigenvalue picked uniformly at random among $\lambda_1^{(n)} \dots \lambda_n^{(n)}$. Let also $p^{(n)}(\lambda)$ denote the distribution of $\lambda^{(n)}$. We are interested in the limiting behaviour of $p^{(n)}(\lambda)$ as $n\to\infty$.

First remark: for most random matrix ensembles, a rescaling of the matrices in n is needed in order to obtain the convergence of $p^{(n)}(\lambda)$.

a) (Square) complex Wishart Ensemble

Let $\underset{(n\times n \text{ matrix})}{H^{(n)}}$ have i.i.d. $N_{\mathbb{C}}(0,1)$ entries and $A^{(n)} = H^{(n)}(H^{(n)})^*$.

What is the first moment of $p^{(n)}(\lambda)$?

$$\int_0^\infty \lambda\, p^{(n)}(\lambda)\, d\lambda = \mathbb{E}(\lambda^{(n)}) = \mathbb{E}\left(\frac{1}{n}(\lambda_1^{(n)} + \dots + \lambda_n^{(n)})\right)$$

$$= \mathbb{E}\left(\frac{1}{n}\operatorname{Tr}(A^{(n)})\right) = \frac{1}{n}\sum_{j,k=1}^{n} \underbrace{\mathbb{E}(|h_{jk}^{(n)}|^2)}_{=1} = \frac{1}{n}\cdot n^2 = n$$

This heuristics indicates that in order to obtain the convergence of at least the first moment of $p^{(n)}(\lambda)$, we need to scale down the matrix $A^{(n)}$ by a factor $\frac{1}{n}$ (note that this heuristics coincides by the way with the "wrong" analysis of page 1, saying that $\frac{1}{n} H^{(n)} (H^{(n)})^*$ converges entrywise to I).

In the case where H is a rectangular $n \times m$ matrix, where $\frac{m}{n} = c$ is a fixed parameter, the same heuristics shows that we should scale down the matrix $A^{(n)}$ by a factor $\frac{1}{n}$.

b) GUE

Let $A^{(n)} = H^{(n)}$ (nxn matrix) have independent entries $h_{jk}^{(n)}$, $j \le k$, with

$h_{jj}^{(n)} \sim N_{\mathbb{R}}(0,1)$, $h_{jk}^{(n)} \sim N_{\mathbb{C}}(0,1)$ and $h_{kj}^{(n)} = \overline{h_{jk}^{(n)}}$

First moment of $p^{(n)}(\lambda)$:

$$\int_{\mathbb{R}} \lambda \, p^{(n)}(\lambda) \, d\lambda = \mathbb{E}(\lambda^{(n)}) = \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n} \lambda_j^{(n)}\right) = \mathbb{E}\left(\frac{1}{n} \operatorname{Tr} A^{(n)}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \underbrace{\mathbb{E}\left(h_{jj}^{(n)}\right)}_{0} = 0$$

So far, it therefore seems that no rescaling is needed, but...

Second moment of $p^{(n)}(\lambda)$:

$$\int_{\mathbb{R}} \lambda^2 p^{(n)}(\lambda)\, d\lambda = \mathbb{E}\left((\lambda^{(n)})^2\right) = \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}(\lambda_j^{(n)})^2\right)$$
$$= \mathbb{E}\Big(\frac{1}{n}\,\mathrm{Tr}\big(\underbrace{(A^{(n)})^2}_{=A^{(n)}(A^{(n)})^*}\big)\Big) = \frac{1}{n}\sum_{j,k=1}^{n} \underbrace{\mathbb{E}\left(|h_{jk}^{(n)}|^2\right)}_{=1} = n$$

According to this heuristics, we therefore need to scale down the matrix $A^{(n)}$ by a factor $\frac{1}{\sqrt{n}}$.

c) CUE

Let $\underset{(n\times n\text{ matrix})}{A^{(n)}}$ be distributed according to the Haar dist. on $U(n)$.

In this case, we have:

$$\int_0^{2\pi} |e^{i\theta}|^2 p^{(n)}(\theta)\, d\theta \overset{(*)}{=} \mathbb{E}\left(|\lambda^{(n)}|^2\right) = \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}|\lambda_j^{(n)}|^2\right)$$
$$= \mathbb{E}\left(\frac{1}{n}\,\mathrm{Tr}\left(A^{(n)}(A^{(n)})^*\right)\right) = \mathbb{E}\left(\frac{1}{n}\,\mathrm{Tr}\, I\right) = 1$$

Therefore, no rescaling is needed. Moreover, $p^{(n)}(\theta) = \frac{1}{2\pi}$ is the uniform distribution on $[0, 2\pi)$ for all n, so the asymptotic analysis is particularly simple!

(*) NB: it is clear that this integral is equal to 1, since $|e^{i\theta}| = 1$ & $p^{(n)}(\theta)$ is a p.d.f.!

Aside: asymptotic analysis of the second order marginal distribution in the CUE

We have: $p^{(n)}(\theta,\varphi) = \frac{1}{4\pi^2}\cdot\frac{n}{n-1}\left(1-\left(\frac{\sin\left(\frac{n(\theta-\varphi)}{2}\right)}{n\sin\left(\frac{\theta-\varphi}{2}\right)}\right)^2\right)$

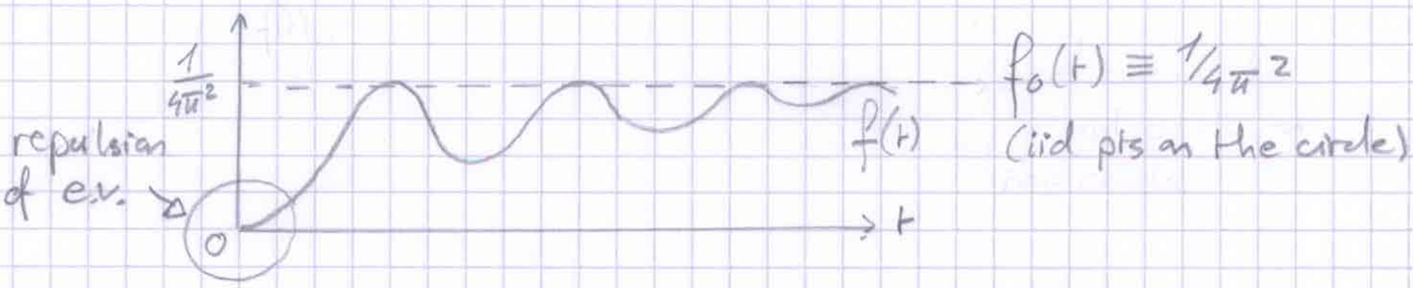
Since there are n eigenvalues on the unit circle of (fixed) length 2π, the natural scaling for the spacings of eigenvalues is of order $\frac{1}{n}$. Let us therefore define $\theta = \varphi + \frac{t}{n}$:

$$p^{(n)}(\varphi+\tfrac{t}{n},\varphi) = \frac{1}{4\pi^2}\cdot\frac{n}{n-1}\left(1-\frac{(\sin(t/2))^2}{(n\sin(t/2n))^2}\right)$$

$$\underset{n\to\infty}{\sim} \frac{1}{4\pi^2}\left(1-\frac{(\sin(t/2))^2}{(t/2)^2}\right) := f(t)$$

"two-point correlation function"

since $n\sin(x/n) \longrightarrow x$ as $n\to\infty$.



A similar behaviour near zero can be obtained for the probability that two neighbouring ev. are separated by a distance t/n (but the analysis is more difficult).

Back to the first order marginal, in the GUE:

We had computed, for $A^{(n)} = H^{(n)}$:

$$p^{(n)}(\lambda) = \sum_{\ell=0}^{n-1} H_\ell(\lambda)^2 e^{-\lambda^2/2} \quad \text{with } H_\ell = \text{Hermite polynomials}$$

Considering now the rescaled matrices $\tilde{A}^{(n)} = \frac{1}{\sqrt{n}} H^{(n)}$, we obtain:

$$\tilde{p}^{(n)}(\lambda) = \sqrt{n}\, p^{(n)}(\sqrt{n}\lambda) = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} H_\ell(\sqrt{n}\lambda)^2 e^{-n\lambda^2/2}$$

Analyzing this expression further requires a good knowledge of Hermite polynomials! By the Christoffel-Darboux formula, we obtain:

$$\tilde{p}^{(n)}(\lambda) = \left(\sqrt{n}\, H_n(\sqrt{n}\lambda)^2 - \sqrt{n+1}\, H_{n-1}(\sqrt{n}\lambda) H_{n+1}(\sqrt{n}\lambda)\right) e^{-n\lambda^2/2}$$

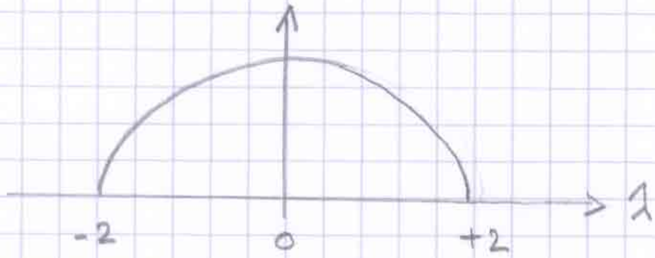
Next, the Plancherel-Rotach formula gives:

$$(2n)^{1/4} H_n(\sqrt{n}\lambda) e^{-n\lambda^2/4} = \text{some } f(\lambda) + O\left(\tfrac{1}{n}\right)$$

... we "therefore" have:

$$\lim_{n\to\infty} \tilde{p}^{(n)}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2} \cdot 1_{|\lambda| \le 2}$$

Wigner's semi-circle law



Remarks

- The above "proof" is quite technical.
- It generalizes to other random matrix ensembles, but requires different formulas for different orthogonal polynomials
- It relies heavily on the fact that both $p^{(n)}(\lambda_1 \ldots \lambda_n)$ and the marginal $p^{(n)}(\lambda)$ are computable at finite n.

We will see that much more powerful techniques allow to strenghten the result in two main directions:

- the result can be shown for more general random matrix ensembles (for which $p^{(n)}(\lambda)$ is unknown at finite n)
- the convergence of $p^{(n)}(\lambda)$ to a limit as $n \to \infty$ is a convergence "in expectation"$^{(*)}$; it can be shown that an "almost sure" convergence also holds for random matrices.

$$(*) \quad \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)})\right) = \mathbb{E}\left(f(\lambda^{(n)})\right)$$

$$= \int_{\mathbb{R}} f(\lambda)\, p^{(n)}(\lambda)\, d\lambda \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(\lambda)\, p(\lambda)\, d\lambda$$