Random matrix theory : lecture 4

Computation of marginals

Square Complex Wishart Ensemble

Let H be a $n\times n$ matrix with iid $\sim N_{\mathbb{C}}(0,1)$ entries

and $\lambda_1 \dots \lambda_n$ be the (non-negative) eigenvalues of $W = HH^*$.

Their joint distribution is given by

$$p(\lambda_1 \dots \lambda_n) = C_n \exp\left(-\sum_{j=1}^{n} \lambda_j\right) \prod_{j<k} (\lambda_k - \lambda_j)^2$$

NB:

In order for the transformation $H = U\Lambda U^*$ to be unique, we need to order the eigenvalues, e.g. as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (\geq 0)$. The above distribution p is therefore defined on this set.

× Note however that p is symmetric under any permutation of $(\lambda_1 \dots \lambda_n)$. The distribution of the unordered eigenvalues of W is therefore given by the same above expression (only the constant

× changes: it is divided by $n!$, but we do not care).

× We will focus on this second object in the following.

× We are interested in computing the first order and second order marginal distributions:

$$\begin{cases} p(\lambda) = \int_{\mathbb{R}_+^{n-1}} p(\lambda, \lambda_2, \dots, \lambda_n)\, d\lambda_2 \dots d\lambda_n \\ p(\lambda, \mu) = \int_{\mathbb{R}_+^{n-2}} p(\lambda, \mu, \lambda_3, \dots, \lambda_n)\, d\lambda_2 \dots d\lambda_n \end{cases}$$

Step 1 : Van der Monde determinant and Laguerre polynomials

In order to marginalize $p(\lambda_1 \dots \lambda_n)$, we need to take care of the term $\prod_{j<k} (\lambda_k - \lambda_j)^2$.

× Van der Monde identity:

$$\prod_{j<k} (\lambda_k - \lambda_j) = \det \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

Laguerre polynomials:

Let us define for $k \geq 0$:

$$L_k(\lambda) = \frac{1}{k!} \cdot e^{\lambda} \cdot \frac{d^k}{d\lambda^k} \left(e^{-\lambda} \lambda^k \right) \quad \lambda \geq 0$$

i.e. $L_0(\lambda) \equiv 1$, $L_1(\lambda) = 1 - \lambda$, $L_2(\lambda) = \frac{1}{2}(\lambda - 2)^2 - 1$ ...

Property A:

L_k is a polynomial of degree k in λ:

$$L_k(\lambda) = \gamma_k \lambda^k + \dots \text{ with } \gamma_k \neq 0 \text{ (actually, } \gamma_k = \frac{(-1)^k}{k!} \text{)}$$

Therefore,

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1}\end{pmatrix} = \underbrace{\left(\prod_{j=0}^{n-1} \frac{1}{\gamma_j}\right)}_{\substack{\text{some constant} \\ c_n \neq 0}} \cdot \underbrace{\det\begin{pmatrix} L_0(\lambda_1) & \cdots & L_0(\lambda_n) \\ \vdots & & \vdots \\ L_{n-1}(\lambda_1) & \cdots & L_{n-1}(\lambda_n)\end{pmatrix}}_{:= \{L_{j-1}(\lambda_k)\}}$$

i.e.

$$\prod_{j<k} (\lambda_k - \lambda_j)^2 = c_n^2 \det\left(\{L_{j-1}(\lambda_k)\}\right)^2$$

$$= c_n^2 \det\left(\{L_{j-1}(\lambda_k)\} \cdot \{L_{j-1}(\lambda_k)\}^T\right)$$

$$= c_n^2 \det\left(\{K(\lambda_j, \lambda_k)\}\right)$$

where $K(\lambda,\mu) := \sum_{l=0}^{n-1} L_l(\lambda) L_l(\mu)$

Remark: for the Real Wishart Ensemble, the square is missing in the Jacobian, so the above trick does not work and the analysis is (much) more complicated.

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So far, our reasoning works if we replace $\{L_k(\lambda)\}$ by any sequence of polynomials. Why then choose the specific Laguerre polynomials? Because of Property B ("orthogonal polynomials")

$$\int_{\mathbb{R}_+} L_k(\lambda) L_l(\lambda) e^{-\lambda} d\lambda = \delta_{kl} = \begin{cases} 1 & \text{if } k=l \\ 0 & \text{otherwise}\end{cases}$$

Step 2: properties of K and Mehta's lemma

Because of property B, we are able to show the following.

Proposition: (o) $K(\mu,\lambda) = K(\lambda,\mu)$

(i) $\int_{\mathbb{R}_+} K(\lambda,\lambda)\, e^{-\lambda}\, d\lambda = n$

(ii) $\int_{\mathbb{R}_+} K(\lambda,\mu)\, K(\mu,\nu)\, e^{-\mu}\, d\mu = K(\lambda,\nu)$

x [i.e. $K(\lambda,\mu)$ is a "self-reproducing kernel"]

Proof: (o) clear

(i) $\sum_{l=0}^{n-1} \underbrace{\int_{\mathbb{R}_+} L_l(\lambda)\, e^{-\lambda}\, d\lambda}_{=1} = n$ ✓

(ii) $\sum_{l,m=0}^{n-1} L_l(\lambda)\, L_m(\nu) \underbrace{\int_{\mathbb{R}_+} L_l(\mu)\, L_m(\mu)\, e^{-\mu}\, d\lambda}_{=\delta_{lm}} = K(\lambda,\nu)$ ✓ #

Recall that

$$p(\lambda_1 .. \lambda_n) = C_n \exp\left(-\sum_{j=1}^{n} \lambda_j\right) \cdot \det\left(\{K(\lambda_j,\lambda_k)\}\right)$$

We have the following lemma from Mehta.

Lemma

x a) $p(\lambda) = \int_{\mathbb{R}_+^{n-1}} p(\lambda,\lambda_2 .. \lambda_n)\, d\lambda_2 .. d\lambda_n = C_{n,1}\, K(\lambda,\lambda)\, e^{-\lambda}$

b) $p(\lambda,\mu) = \int_{\mathbb{R}_+^{n-2}} p(\lambda,\mu,\lambda_3 .. \lambda_n)\, d\lambda_3 .. d\lambda_n$

x $\qquad = C_{n,2} \left(K(\lambda,\lambda) K(\mu,\mu) - K(\lambda,\mu)^2 \right) e^{-(\lambda+\mu)}$

<u>Sketch of proof</u>: $(n=2\,!)$

b) $p(\lambda_1,\lambda_2) = c_2\, e^{-(\lambda_1+\lambda_2)} \det\begin{pmatrix} k(\lambda_1,\lambda_1) & k(\lambda_1,\lambda_2) \\ k(\lambda_2,\lambda_1) & k(\lambda_2,\lambda_2)\end{pmatrix}$ ✓

a) $$p(\lambda_1) = \int_{\mathbb{R}_+} p(\lambda_1,\lambda_2)\, d\lambda_2$$

$$= c_2\, e^{-\lambda_1}\Big(k(\lambda_1,\lambda_1) \overbrace{\int_{\mathbb{R}_+} k(\lambda_2,\lambda_2)\, e^{-\lambda_2}\, d\lambda_2}^{=2 \text{ by (i)}}$$
$$- \underbrace{\int_{\mathbb{R}_+} k(\lambda_1,\lambda_2)\, k(\lambda_2,\lambda_1)\, e^{-\lambda_2}\, d\lambda_2}_{=k(\lambda_1,\lambda_1) \text{ by (ii)}} \Big)$$

$$= c_2\, e^{-\lambda_1}\, k(\lambda_1,\lambda_1) \;\checkmark$$

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<u>Last step</u>: determination of the constants $c_{n,1}$ & $c_{n,2}$

- $\displaystyle\int_{\mathbb{R}_+} p(\lambda)\, d\lambda = 1 \implies c_{n,1} \underbrace{\int_{\mathbb{R}_+} k(\lambda,\lambda)\, e^{-\lambda}\, d\lambda}_{=n \text{ by (i)}} = 1$

  i.e. $c_{n,1} = \frac{1}{n}$

- $\displaystyle\int_{\mathbb{R}_+} p(\lambda,\mu)\, d\mu = p(\lambda)$

  $$\implies c_{n,2}\big(\underbrace{k(\lambda,\lambda)\cdot n}_{\text{by (i)}} - \underbrace{k(\lambda,\lambda)}_{\text{by (ii)}}\big)\, e^{-\lambda} = c_{n,1}\, k(\lambda,\lambda)\, e^{-\lambda}$$

  i.e. $c_{n,2} = \frac{c_{n,1}}{n-1} = \frac{1}{n(n-1)}$

<u>Finally</u>:
$$\begin{cases} p(\lambda) = \frac{1}{n}\, k(\lambda,\lambda)\, e^{-\lambda} = \frac{1}{n}\sum_{\ell=0}^{n-1} L_\ell(\lambda)^2 e^{-\lambda} \\ p(\lambda,\mu) = \frac{1}{n(n-1)}\big(k(\lambda,\lambda)\, k(\mu,\mu) - k(\lambda,\mu)^2\big)\, e^{-(\lambda+\mu)} \\ \text{with } k(\lambda,\mu) = \sum_{\ell=0}^{n-1} L_\ell(\lambda)\, L_\ell(\mu) \end{cases}$$

Marginals for the GUE

joint eigenvalue distribution: ($\lambda_j \in \mathbb{R}$)

$$p(\lambda_1 \ldots \lambda_n) = C_n \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \lambda_j^2\right) \cdot \prod_{j<k} (\lambda_k - \lambda_j)^2$$

$\Rightarrow$ the analysis is totally similar; the only difference comes from the fact that one has to consider Hermite polynomials (instead of Laguerre pol.)

$$H_k(\lambda) = c_k e^{\lambda^2/2} \frac{d^k}{d\lambda^k}\left(e^{-\lambda^2/2}\right) \qquad \lambda \in \mathbb{R}$$

that satisfy:

$$\int_{\mathbb{R}} H_k(\lambda) H_\ell(\lambda) e^{-\lambda^2/2} d\lambda = \delta_{k\ell}$$

We obtain that

$$\begin{cases} p(\lambda) = \frac{1}{n} \sum_{\ell=0}^{n-1} H_\ell(\lambda)^2 e^{-\lambda^2/2} \\ p(\lambda,\mu) = \frac{1}{n(n-1)} \left(K(\lambda,\lambda) K(\mu,\mu) - K(\lambda,\mu)^2\right) e^{-\frac{\lambda^2+\mu^2}{2}} \\ \text{with } K(\lambda,\mu) = \sum_{\ell=0}^{n-1} H_\ell(\lambda) H_\ell(\mu) \end{cases}$$

<u>NB</u>: The method therefore generalizes to

$$p(\lambda_1 \ldots \lambda_n) = C_n \exp\left(-\sum_{j=1}^{n} V(\lambda_j)\right) \cdot \prod_{j<k} (\lambda_k - \lambda_j)^2$$

as soon as there exist polynomials satisfying

$$\int_{\mathbb{R}} P_k(\lambda) P_\ell(\lambda) \underbrace{e^{-V(\lambda)}}_{\text{weight}} d\lambda = \delta_{k\ell}$$

21.3.07
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RMT: linear algebra reminder

× **Eigenvalues and singular values**

- **Fact 1**: Let H be a $n \times n$ complex Hermitian matrix.

There exist U $n\times n$ unitary and $\Lambda = \mathrm{diag}(\lambda_1 \ldots \lambda_n)$

such that $H = U \Lambda U^*$. This implies: $HU = U\Lambda$

× i.e. λ_k = eigenvalue of H with corr. eigenvector $u^{(k)} = (u_{jk})$ (column of U)

- **Fact 2**: Let H be a $n \times m$ complex matrix.

There exist U $n\times n$ unitary, V $m\times m$ unitary and

× $\Sigma = \mathrm{pseudodiag}(\sigma_1 \ldots \sigma_r)$ with $r = \min(n,m)$,
($n\times m$ matrix with off-diagonal entries $=0$)

such that $H = U \Sigma V^*$.

$\sigma_1 \ldots \sigma_r$ are called the singular values of H.

This implies:

× a) $HH^* U = U \Sigma \Sigma^*$, i.e. the columns of U are the eigenvectors of HH^*, with non-zero eigenvalues $\sigma_1^2 \ldots \sigma_r^2$.

× b) $H^* H V = V \Sigma^* \Sigma$, i.e. the columns of V are the eigenvectors of H^*H, again with non-zero eigenvalues $\sigma_1^2 \ldots \sigma_r^2$.

The squares of the singular values of H are therefore the non-zero eigenvalues of HH^* (or H^*H).

<u>Relation between the eigenvalues and the singular values:</u>

(valid only for square matrices H !)

We know that any $n \times n$ complex matrix H also has n eigenvalues $\lambda_1 .. \lambda_n$. What is the relation between these and $\sigma_1 .. \sigma_n$, the singular values of H?

- In general, there is <u>no</u> specific relation, except some inequalities, such as $\max_k |\lambda_k| \leq \max_k \sigma_k$. (*)

- If H is Hermitian, then $HH^* = H^*H = H^2$, so $U = V$ and $\sigma_j^2 = \lambda_j^2$ (provided that they are ordered accordingly), so $\sigma_j = |\lambda_j|$ [recall that $\sigma_j \geq 0$, and $\lambda_j \in \mathbb{R}$ when $H = H^*$].

- <u>Ex</u>: $H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \lambda_1 = \lambda_2 = 1$

$$HH^* = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \sigma_1^2 = \frac{3+\sqrt{5}}{2}, \; \sigma_2 = \frac{3-\sqrt{5}}{2}$$

(*) and of course: $\left| \prod_{k=1}^{n} \lambda_k \right| = |\det H| = \sqrt{\det(HH^*)} = \prod_{k=1}^{n} \sigma_k$