Random matrix theory: Lecture 25

Recall: \( T = \bigoplus_{n=0}^{\infty} T^0_n \), with \( T^0 = \mathbb{C}^2 \), basis \( (e_1, e_2) \)

- \( A = \{ a: T \rightarrow T \text{ linear bounded operator} \} \)
- \( \phi(a) = \langle 1, a 1 \rangle \)
- \( a_i, a_i^* \): creation and annihilation operators
- \( A_i = a_i + a_i^* \): distributed according to the semi-circle distribution: \( R_{A_i}(z) = \frac{1}{\pi} \sqrt{4 - z^2} \)

- \( A_1 \) and \( A_2 \) freely independent

Comment 1

\[ R_{A_1 A_2}(z) = R_{A_1}(z) + R_{A_2}(z) = 2 + 2 = 4 \]

ie. \( A_1 A_2 \) is again distributed according to the semi-circle distribution (with a different variance).

ie. the semi-circle distribution plays the same role in free probability as the Gaussian distribution in classical probability. The analogy goes on with the following theorem (turn the page)...
**Free central limit theorem**

Let \( a_1, \ldots, a_n, \ldots \) be a sequence of freely independent random variables, identically distributed and such that

\( \varphi(a_1) = 0 \) and \( \varphi(a_1^2) = 1 \).

Let \( \mu_n \) be the distribution of \( \frac{1}{\sqrt{n}} (a_1 + \ldots + a_n) \).

Then \( \mu_n \) converges to the semi-circle distribution as \( n \to \infty \).

(i.e. \( R_{\mu_n}(z) \xrightarrow{n \to \infty} \pi(z) \forall z \in \mathbb{C} \))

**Proof idea:**

1) For \( c \) a constant and \( A \) a random variable, \( R_{cA}(z) = cR_A(cz) \):

\[
\begin{align*}
q_{cA}(z) &= \varphi((cA - zI)^{-1}) = \frac{1}{c} \varphi((A - \frac{z}{c}I)^{-1}) = \frac{1}{c} q_A\left(\frac{z}{c}\right) \\
so \quad q_{cA}^{-1}(z) &= c q_A^{-1}(cz) \quad \left[ \frac{1}{c} q_A\left(\frac{z}{c}\right) = 5 \right] \quad c q_A^{-1}(cz) = z
\end{align*}
\]

and \( R_{cA}(z) = q_{cA}^{-1}(z) - \frac{1}{z} = cR_A(cz) \).

2) \( R_{\frac{1}{\sqrt{n}}(a_1 + \ldots + a_n)}(z) = \frac{1}{\sqrt{n}} R_{a_1 + \ldots + a_n}\left(\frac{3}{\sqrt{n}}\right) \) by (1)

(by free indep.) \( = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} R_{a_j}\left(\frac{3}{\sqrt{n}}\right) = \sqrt{n} R_{a_1}\left(\frac{3}{\sqrt{n}}\right) \)

3) expansion: \( R_{a_1}(z) = \sum_{k=0}^\infty c_k z^k \)

\( c_0 = \varphi(a_1) = 0 \)

\( c_1 = \varphi(a_1^2) - \varphi(a_1)^2 = 1 \)

\( \Rightarrow R_{a_1}(z) = z + o(z) \)

i.e. \( R_{\frac{1}{\sqrt{n}}(a_1 + \ldots + a_n)}(z) = \sqrt{n}\left(\frac{3}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) = z + o(1) \)
Comment 2

The fact that $A_n = q_1 \ast q_n$ is distributed according to the semi-circle distribution can be put in relation with the following:

- Let $A^{(n)} = \frac{1}{n} \sum \text{ (deterministic) matrices }$ (with $n \to \infty$)

$$Q(A) := q_{n1}$$

$$A^{(n)} := \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

deterministic Toeplitz matrix

Then

$$\lim_{n \to \infty} Q((A^{(n)})^k) = \begin{cases} \frac{1}{k+1} & \text{if } k = 2l \\ 0 & \text{if } k = 2l + 1 \end{cases}$$

i.e. as $n \to \infty$, $A^{(n)}$ is distributed according to the semi-circle distribution (with respect to expectation $Q$).

"Proof"

- Note that with such an expectation $Q$, changing only two entries in the Toeplitz matrix changes drastically the limit. Let $B^{(n)} := \begin{pmatrix} 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$ [circulant matrix!]

Then

$$\lim_{n \to \infty} Q((B^{(n)})^k) = \begin{cases} \frac{1}{k} & \text{if } k = 2l \\ 0 & \text{otherwise} \end{cases}$$

Proof: use $b^{(n)}_{11} = \frac{1}{n} \text{Tr}(B^{(n)})$
Free multiplicative convolution and S-transform

Let $A$ be a non-commutative random variable such that $\varphi(A) \neq 0$.

Let $\left( \varphi_A(z) : = \sum_{k \geq 1} \varphi(A^k) z^k \right) = \varphi((I-zA)^{-1})$

\[ = -\frac{1}{z} \varphi_A\left(\frac{1}{z}\right) - 1 \]

\[ S_A(z) : = \frac{z+1}{z} \varphi_A^{-1}(z) \] \begin{itemize}
  \item \text{inverse function, Stieltjes transform}
\end{itemize}

**Proposition:**

If $A_1, A_2$ are freely independent and s.t. $\varphi(A_1) \neq 0, \varphi(A_2) \neq 0$, then

$S_{A_1 A_2}(z) = S_{A_1}(z) S_{A_2}(z)$

and the distribution of $A_1 A_2$ is called the free multiplicative convolution and is denoted as $\mathcal{M}_{A_1 A_2} = \mathcal{M}_{A_1} \boxtimes \mathcal{M}_{A_2}$.

**Proof idea:**

Same technique as for the R-transform:

- Let $A_1 = (I + a_1) \cdot P(a_1^{-1})$ with $p(x)$ some polynomial s.t. $p(0) \neq 0$.

- Then $S_{A_1}(z) = \frac{1}{p(z)}$; similarly $S_{A_2}(z) = \frac{1}{q(z)}$

- and $S_{A_1 A_2}(z) = \frac{1}{p(z) q(z)}$

- (*) Example: $A \sim \text{quarter circle}$, i.e. $2q_A(z)^2 + 2q_A(z) + 1 = 0$

  $\Rightarrow z (q_A(z) + 1)^2 = q_A(z) \Rightarrow S_A(z) = \frac{1}{z+1}$
Application to random matrices

Let $A^{(n)}$, $B^{(n)}$ be $n \times n$ real symmetric independent random matrices with limiting eigenvalue distributions $\nu_A^{(n)}$, $\nu_B^{(n)}$. Let $V^{(n)}$ be orthogonal (Haar dist.) and independent of both $A^{(n)}$ and $B^{(n)}$.

Then $A^{(n)}$, $V^{(n)}B^{(n)}(V^{(n)})^T$ are asymptotically free, so the limiting eigenvalue distribution of $A^{(n)}V^{(n)}B^{(n)}(V^{(n)})^T$ is given by $\nu_{AB} = \nu_A \boxtimes \nu_B$ and can be computed via the $S$-transform.

\[ (*) \text{ such that } \int_{\mathbb{R}} x \, d\nu_A(x) \neq 0 \text{ and } \int_{\mathbb{R}} x \, d\nu_B(x) \neq 0 \]