Random matrix theory: Lecture 23

Reminder

- $A, B$ free non-commutative random variables,

with corresponding distributions $\mathcal{M}_A = (m_A^A(k), k \geq 0)$, $\mathcal{M}_B = (m_B^B(k), k \geq 0)$

$\Rightarrow$ the distribution of $A + B$ given by $\mathcal{M}_{A+B} = (m_{A+B}^A(k), k \geq 0)$

is the free additive convolution of $\mathcal{M}_A$ and $\mathcal{M}_B$ (denoted as $\mathcal{M}_{A+B} = \mathcal{M}_A \boxplus \mathcal{M}_B$), which can be computed using the R-transform:

$$R_{A+B}(z) = R_A(z) + R_B(z).$$

- $A^{(n)}, B^{(n)}$ independent random matrices with limiting eigenvalue distributions $\mathcal{M}_A, \mathcal{M}_B$

$\Rightarrow$ let $V^{(n)}$ be orthogonal & independent of both $A^{(n)}$ and $B^{(n)}$ (Haar distributed),

then $A^{(n)}$ & $V^{(n)}B^{(n)}(V^{(n)})^T$ are asymptotically free

so the limiting eigenvalue distribution of $A^{(n)} + V^{(n)}B^{(n)}(V^{(n)})^T$

is given by $\mathcal{M}_{A+B} = \mathcal{M}_A \boxplus \mathcal{M}_B$ and can be computed via the R-transform.

Today's program: [ref: Haagerup, Thomaßjönsen]

- construction of free random variables
- proof of the R-transform additivity rule.
- free multiplicative convolution
Construction of free random variables

Preliminary:

- An $n \times n$ matrix $A$ is a linear application $\mathbb{C}^n \to \mathbb{C}^n$.
  It is entirely determined by its action on the elements of any basis of $\mathbb{C}^n$ (e.g. $e_1, \ldots, e_n$).

- More generally, let $T$ be a Hilbert space with a countable basis (possibly infinite) and $A$ be a linear and bounded operator $T \to T$.
  Then $A$ is entirely determined by its action on the basis elements of $T$.

Example:

Let $H := \mathbb{C}^2$, with basis $(e_1, e_2)$

$H^\otimes n$ := tensor product, with basis $(e_1 \otimes \ldots \otimes e_i, i = n \in \mathbb{N}, 2)$

(with the convention $H^\otimes 0 := \mathbb{C} : 1$ with basis $(1)$)

$T := \bigoplus_{n \geq 0} H^\otimes n$ ; $T$ has the following countable basis:

$(1, e_1, e_2, e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_1 \otimes e_1, \ldots)$

Interpretation : elements of $T$ represent a physical system in a given state; $n$ represents the number of particles, each of which is either in state $e_1$ or state $e_2$; $1$ (corresponding to $n=0$) represents the empty state.
In the following, we will consider non-commutative random variables as linear and bounded operators $a: \mathcal{L} \rightarrow \mathcal{L}$.

We also define the expectation: $\varphi(a) := \langle 1, a \rangle$

(cf. $\varphi(A) = a_{11}$ in the matrix case)

Note: the scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{L}$ is "defined" by

\[
\langle e_{i_1} \otimes \ldots \otimes e_{i_m}, e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle = \delta_{m,n} \delta_{i_1=j_1} \ldots \delta_{i_m=j_n}
\]

0 otherwise

The corresponding norm on $\mathcal{L}$ is defined by

\[
\|a\| := \sqrt{\langle a, a \rangle}
\]

and $a: \mathcal{L} \rightarrow \mathcal{L}$ is bounded if $\exists k > 0$ such that

\[
\|a h\| \leq k \|h\| \quad \forall h \in \mathcal{L}
\]

(Note that on an n x n matrix $A$ always satisfies this condition.)

Examples of non-commutative random variables on $\mathcal{L}$:

- "creation operator":

\[
a_i (e_{i_1} \otimes \ldots \otimes e_{i_n}) := e_{i_1} \otimes \ldots \otimes e_{i_n} \quad i = 1, 2
\]

one more particle in state $i$

- "annihilation operator": one less particle in state $i$

\[
a_i^* (e_{i_1} \otimes \ldots \otimes e_{i_n}) := \delta_{i_1=1} e_{i_2} \otimes \ldots \otimes e_{i_n} \quad \text{if} \quad i_1 = 1
\]

\[
0 \quad \text{otherwise}
\]

cancellation: $a_i^* 1 = 0$ ...
or even nothing...

Basic properties:

1. \( \forall h_1, h_2 \in \mathbb{C}, \quad \langle a_i^* h_1, h_2 \rangle = \langle h_1, a_i h_2 \rangle; \quad a_i^* : \text{dual of } a_i \)

**Proof:** Check this for basis elements:

\[
\langle a_i^* e_{i_1} \otimes \ldots \otimes e_{i_m}, e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle = 
\]

\[
= \langle e_{i_1} \otimes \ldots \otimes e_{i_m}, a_i e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle 
\]

i.e.

\[
\delta_{i_1 i_j} \langle e_{i_1} \otimes \ldots \otimes e_{i_m}, e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle 
\]

Both sides are equal to 1 if \( n = m+1, \quad i = i_1, i_2 = j_1 \ldots, i_m = j_n \) #

2. \( a_i^* a_i = I \): \( a_i^* a_i e_{i_1} \otimes \ldots \otimes e_{i_m} = a_i^* a_i e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_m} = e_{i_1} \otimes \ldots \otimes e_{i_m} \) #

3. \( a_i^* a_j = 0 \) if \( i \neq j \): clear, since \( a_j \) creates a particle in state \( j \); so \( a_i^* \) annihilates the whole system #

**Proposition**

Let \( p(x, y) \), \( q(x, y) \) be any two polynomials. Then \( p(a_1, a_1^*) \) and \( q(a_2, a_2^*) \) are freely independent.

**Proof:**

- Since \( a_i^* a_i = I \), \( p(a_1, a_1^*) \) may always be written as a sum of terms of the form \( a_1^k (a_1^*)^k \) with \( k \geq 0 \) (and the identity term). Same is true for \( q(a_2, a_2^*) \).
Let us therefore check the free independence of terms of the form $a_i^k (a_i^*)^l$ and $b_i^k (b_i^*)^l$ with $k+l>0$:

First note that:

$$\langle a_i^k (a_i^*)^l \rangle = \langle 1, a_i^k (a_i^*)^l \rangle = \langle (a_i^*)^k 1, (a_i^*)^l \rangle = 0 \quad \text{since at least one } l>0$$

(recall that $a_i^*$ annihilates operator)

There remains to show therefore that:

$$\phi(a_i^k (a_i^*)^l, a_2^k (a_2^*)^l, \ldots, a_i^k (a_i^*)^l, a_2^k (a_2^*)^l) = 0$$

Suppose it is not the case; then $k_1 = 0$, otherwise

$$\langle a_i^{k_1} \ldots \rangle = \langle 1, a_i^{k_1} \ldots 1 \rangle = \langle (a_i^*)^{k_1} 1, \ldots 1 \rangle = 0$$

Now, $k_1 + l_1 > 0 \Rightarrow l_1 > 0$, so $k_2 = 0$, otherwise

$$(a_i^*)^{l_1} a_2^{k_2} h = 0 \quad \forall h \Rightarrow \phi(\ldots) = 0$$

Now, $k_2 + l_2 > 0 \Rightarrow l_2 > 0$, so $k_3 = 0$ etc...

Finally, (by induction) $k_1 = k_2 = \ldots = k_m = 0$ and

$$\phi(\ldots) = \phi((a_1^*)^{l_1} (a_2^*)^{l_2} \ldots (a_i^*)^{l_{i-1}} (a_2^*)^{l_{i-1}}) = 0$$

contradiction.

So $\phi(\ldots)$ must be equal to zero,

and therefore $\rho(a_1, a_1^*)$ and $\rho(a_2, a_2^*)$

are indeed freely independent.
Now that we know that free non-commutative random variables indeed exist, let us derive
the additivity rule for the \( R \)-transform.

**Proposition**

Let \( p(x) \) be a polynomial and \( A_n = a_n + p(a_n^*) \).

Then \( R_{A_n}(z) = p(z) \).

*Example: \( A_n = a_n + a_n^* \rightarrow R(z) = \overline{z} \) in semi-circle dist.*

Recall: the distribution of \( A_n \) is given by \( M_{k}^{A_n} = \varphi(A_n^{k}) \)

- its Stieltjes transform is given by \( g_{A_n}(z) = \varphi((A_n - zI)^{-1}) \).
- its \( R \)-transform is given by \( R_{A_n}(z) = g_{A_n}^{-1}(-z) - \frac{1}{z} \).

**Proof:**

Let \( W_2 = \sum_{n \geq 2} z^n e^{\Theta_n} = 1 + z e_1 + z^2 e_2 e_1 + \ldots \in \mathcal{T}(\mathbb{E}_{1,c}) \)

Then \( a_n^* W_2 = \sum_{n \geq 2} z^n e_1^{\Theta(n-1)} = z \sum_{n \geq 2} z^n e_1^{\Theta n} = z W_2 \)

so \( p(a_n^*) W_2 = p(z) W_2 \)

Also, \( a_n W_2 = \sum_{n \geq 2} z^n e_1^{\Theta(n+1)} = \frac{1}{z} \sum_{n \geq 1} z^n e_1^{\Theta n} = \frac{1}{z} (W_2 - 1) \)

and \( A_n W_2 = \frac{1}{z} (W_2 - 1) + p(z) W_2 = (\frac{1}{z} + p(z)) W_2 - \frac{1}{z} I \)

i.e. \( (A_n - \frac{1}{z} - p(z)) W_2 = -\frac{1}{z} I \) or \( (A_n - \frac{1}{z} - p(z))^{-1} I = -z W_2 \)

\( \varphi((A_n - \frac{1}{z} - p(z))^{-1}) = -z \) \( <1, W_2> = -z \)

\( = g_{A_n}^{-1}(-z) \) for \( g_{A_n}^{-1}(-z) = \frac{1}{z} + p(z) \) and \( R(z) = p(z) \).
Let $p(x), q(x)$ be any two polynomials (may be extended to $p, q$ analytic functions) and $A_1 = a_1 + p(a_1^*)$, $A_2 = a_2 + q(a_2^*)$.

Then $A_1$ and $A_2$ are freely independent and $R_{A_1 + A_2}(z) = R_{A_1}(z) + R_{A_2}(z)$.

**Proof:**

One has to prove that $R_{A_1 + A_2}(z) = p(z) + q(z)$.

Let $e_2 = \sum_{n \geq 0} z^n (e_1 + e_2)^{\otimes n}$, $|z| < \frac{1}{2}$

Then $(a_1 + a_2) e_2 = \sum_{n \geq 0} z^n (e_1 + e_2)^{\otimes (n+1)} = \frac{1}{2} (p_2 - 1)$

$a_1^* e_2 = \sum_{n \geq 0} z^n a_1^* (a_1 + a_2)^{\otimes n} = 1$

$= \sum_{n \geq 1} z^n (a_1 + a_2)^{\otimes (n+1)} = 2 e_2$

Similarly, $p(a_1^*) e_2 = p(z) e_2$; $a_2^* e_2 = 2 e_2$, $q(a_2^*) e_2 = q(z) e_2$.

So $(A_1 + A_2) e_2 = \frac{1}{2} (p_2 - 1) + (p(z) + q(z)) e_2$

$(A_1 + A_2 - \frac{1}{2} - p(z) - q(z)) e_2 = -\frac{1}{2} 1$

$(A_1 + A_2 - \frac{1}{2} - p(z) - q(z))^{-1} 1 = -2 p_2$

$QA_1 + A_2 (\frac{1}{2} + p(z) + q(z)) = -2$ since $\langle 1, p_2 \rangle = 1$

i.e. $qa^{-1}_{A_1 + A_2}(-2) = \frac{1}{2} + p(z) + q(z)$

and $R_{A_1 + A_2}(z) = qa^{-1}_{A_1 + A_2}(-2) - \frac{1}{2} = p(z) + q(z)$.