Random matrix theory: lecture 15

Marčenko-Pastur's theorem

Let H be a $n \times n$ real or complex random matrix such that

(i) $\{h_{jk},\ j,k = 1..n\}$ are iid random variables

(ii) h_{11} is a bounded random variable (i.e. $|h_{11}(\omega)| \leq C\ \forall \omega$)

(iii) $\mathbb{E}(h_{11}) = 0$, $\mathbb{E}(|h_{11}|^2) = 1$

and let $W^{(n)} := \frac{1}{n} H H^*$, $\lambda_1^{(n)} .. \lambda_n^{(n)}$ be the eigenvalues of $W^{(n)}$

($\underline{NB}$: $\lambda_j^{(n)} \geq 0$)

Remarks:

- assumption (ii) may be dropped without affecting the result
- the assumption that $\mathbb{E}(h_{11}) = 0$ may also be dropped (see lecture 17) and one may also assume a variance different than 1; this changes slightly the limiting distribution.

Theorem (Marčenko-Pastur 1967, Bai-Silverstein 1995)

Under assumptions (i)-(iii),

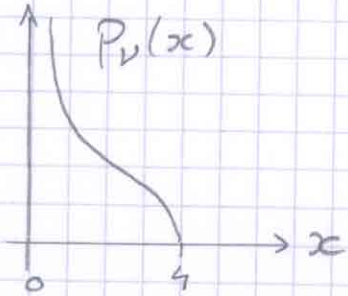
$$F_n(t) := \frac{1}{n} \# \{ j : \lambda_j^{(n)} \leq t \} \xrightarrow[n \to \infty]{} \int_0^{\infty} p_\nu(x)\, dx \quad \text{a.s. } \forall t \geq 0$$

where $p_\nu(x) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}}\ 1_{0 < x < 4}$

"quarter-circle" distribution

Remarks:

- Marčenko-Pastur's theorem is more general: see lecture 16
- The "quarter-circle" distribution p_ν looks like:



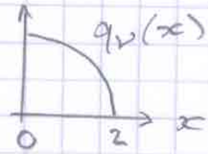
and is therefore not a quater-circle ...

But let us consider instead the limiting distribution of the <u>singular values</u> of $H^{(n)} := \frac{1}{\sqrt{n}} H$: since these are the square roots of the $\lambda_j^{(n)}$, their limiting distribution is:

$$q_\nu(x) = p_\nu(x^2) \cdot 2x = \frac{1}{\pi}\sqrt{4-x^2}\, 1_{0<x<2}$$

(Jacobian of $x \mapsto x^2$)

which is indeed a quater-circle:



- Note that the singular values of $H^{(n)}$, which is non-Hermitian, have a priori no relation with its eigenvalues. It can be shown however that the limiting eigenvalue distribution of $H^{(n)}$ is the uniform distribution on the disc of radius 2 in the complex plane!

<u>Proof</u> (main ideas)

This time, we are going to use the Stieltjes transform:

- $$g_n(z) := \int_0^\infty \frac{1}{x-z}\, dF_n(x) = \frac{1}{n}\sum_{j=1}^n \frac{1}{\lambda_j^{(n)} - z}$$
  $$= \frac{1}{n}\operatorname{Tr}\left(\left(W^{(n)} - zI\right)^{-1}\right)$$

- $$g_\nu(z) := \int_0^\infty \frac{1}{x-z}\, p_\nu(x)\, dx = \frac{1}{\pi}\int_0^4 \frac{1}{x-z}\sqrt{\frac{1}{x} - \frac{1}{4}}\, dx$$
  $$= \dots = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{z}} \quad \text{for } \operatorname{Im} z > 0$$

<u>NB</u>: the reverse formula $p_\nu(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi}\operatorname{Im}\left(g_\nu(x + i\varepsilon)\right)$ is actually easier to check!

More importantly, note that $g_\nu(z)$ is solution of the second order equation:

$$z \cdot g(z)^2 + z \cdot g(z) + 1 = 0$$

- We are now going to prove that:

$$\lim_{n\to\infty} \sup_{z \in K} \left| g_n(z) - g_\nu(z) \right| = 0 \quad \text{a.s.,} \quad \forall K = [-A, A] \times [b, B], \quad A > 0,\ B > b > 0$$

$\forall z \in \mathbb{C}$ s.t. $\operatorname{Im} z > 0$

(we do not show uniform convergence here)

which implies the result.

First notice that $W^{(n)} = \frac{1}{n} H H^* = \frac{1}{n} \sum_{k=1}^{n} \underbrace{h_k h_k^*}_{n\times n \text{ matrix}}$,

where $h_k = k^{th}$ column of H.

Let $W_k^{(n)} := W^{(n)} - \frac{1}{n} h_k h_k^* = \frac{1}{n} \sum_{\substack{l=1 \\ l\neq k}}^{n} h_l h_l^*$.

and $G^{(n)}(z) = \left(W^{(n)} - zI\right)^{-1}$, $G_k^{(n)}(z) = \left(W_k^{(n)} - zI\right)^{-1}$.

Note that $g_n(z) = \frac{1}{n} \operatorname{Tr} G^{(n)}(z)$; we are interested in finding a limiting equation for $g_n(z)$.

<u>Lemma 1</u>

$$\frac{1}{n} h_k^* G^{(n)}(z) h_k = \frac{\frac{1}{n} h_k^* G_k^{(n)}(z) h_k}{1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k}$$

<u>Proof</u>

$$h_k^* G_k^{(n)}(z) \overbrace{\left(W^{(n)} - zI\right)}^{= G^{(n)}(z)^{-1}} = h_k^* G_k^{(n)}(z) \left(W_k^{(n)} - zI + \frac{1}{n} h_k h_k^*\right)$$

$$= h_k^* + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k h_k^* = \left(1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k\right) h_k^*$$

$$\Rightarrow h_k^* G_k^{(n)}(z) = \left(1 + \underbrace{\frac{1}{n} h_k^* G_k^{(n)}(z) h_k}_{\text{scalar}}\right) h_k^* G^{(n)}(z) \qquad (*)$$

so $h_k^* G_k^{(n)}(z) h_k = \left(1 + \frac{1}{n} h_k^* G_k^{(n)} h_k\right) h_k^* G^{(n)}(z) h_k$. #

<u>Lemma 2</u>

$$g_n(z) = \frac{1}{n} \operatorname{Tr} G^{(n)}(z) = -\frac{1}{nz} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k}$$

Proof

$$1 = \frac{1}{n}\mathrm{Tr}(I) = \frac{1}{n}\mathrm{Tr}\left((W^{(n)} - zI)\,G^{(n)}(z)\right)$$

$$= \frac{1}{n}\mathrm{Tr}\left(\frac{1}{n}\sum_{k=1}^{n} h_k h_k^* G^{(n)}(z) - z\,G^{(n)}(z)\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{n} h_k^* G^{(n)}(z)\, h_k - z\,g_n(z)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{\frac{1}{n} h_k^* G_k^{(n)}(z)\, h_k}{1+\frac{1}{n} h_k^* G_k^{(n)}(z) h_k} - z\,g_n(z) \qquad \text{by Lemma 1}$$

$$\Rightarrow z\,g_n(z) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\frac{1}{n} h_k^* G_k^{(n)}(z)\, h_k}{1+\frac{1}{n} h_k^* G_k^{(n)}(z) h_k} - 1\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{1}{n} h_k^* G_k^{(n)}(z)\, h_k} \qquad \#$$

NB: this formula holds for any matrix of the form $W^{(n)} = \frac{1}{n} HH^*$.

Lemma 3

$$\frac{1}{n} h_k^* G_k^{(n)}(z)\, h_k \cong \frac{1}{n}\mathrm{Tr}\left(G_k^{(n)}(z)\right) \quad \text{as } n\to\infty$$

more precisely:

$$\mathbb{P}\left(\left|\frac{1}{n} h_k^* G_k^{(n)}(z)\, h_k - \frac{1}{n}\mathrm{Tr}\left(G_k^{(n)}(z)\right)\right| > a\right) \le \frac{C(a)}{n^2}$$

which implies, by the Borel-Cantelli lemma,

that the difference converges to zero a.s. as $n\to\infty$.

Proof idea

We only check here that expectations are equal:

$$\mathbb{E}\left(\frac{1}{n} h_k^* G_k^{(n)}(z)\, h_k\right) = \frac{1}{n}\sum_{j,\ell=1}^{n} \mathbb{E}\left(\overline{h_{jk}}\,(W_k^{(n)} - zI)^{-1}_{j\ell}\, h_{\ell k}\right)$$

Since $W_k^{(n)}$ does not "contain" the k^{th} column of H, it is independent of both $\overline{h_{jk}}$ and $h_{\ell k}$ (assumption (i)), so

$$\ldots = \frac{1}{n}\sum_{j,\ell=1}^{n} \underbrace{\mathbb{E}\left(\overline{h_{jk}}\, h_{\ell k}\right)}_{=\,\delta_{j\ell} \text{ by assumption (iii)}} \mathbb{E}\left(\left(W_k^{(n)} - zI\right)^{-1}_{j\ell}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}\left(\left(W_k^{(n)} - zI\right)^{-1}_{jj}\right) = \mathbb{E}\left(\frac{1}{n}\operatorname{Tr} G_k^{(n)}(z)\right) \quad \text{"}\#\text{"}$$

NB: assumption (ii) is needed in order to prove that a.s. convergence holds, but there is also a way to get rid of it.

Lemma 4

$$\left|\frac{1}{n}\cdot \operatorname{Tr} G_k^{(n)}(z) - \frac{1}{n}\cdot \operatorname{Tr} G^{(n)}(z)\right| \leq \frac{1}{n\cdot |\operatorname{Im} z|} \quad \forall z \text{ s.t. } \operatorname{Im} z \neq 0$$

Proof: homework

NB: this lemma also holds for any matrix of the form $W^{(n)} = \frac{1}{n} HH^*$.

Conclusion of the proof of the theorem:

By Lemmas 2, 3 & 4, we have:

$$g_n(z) = -\frac{1}{nz} \sum_{k=1}^{n} \frac{1}{1+\frac{1}{n} h_k^* G_k^{(n)}(z) h_k}$$

$$\underset{n\to\infty}{\simeq} -\frac{1}{nz} \sum_{k=1}^{n} \frac{1}{1+g_n(z)} = -\frac{1}{z} \frac{1}{(1+g_n(z))}$$

so $\quad z\, g_n(z)^2 + z\, g_n(z) + 1 \underset{n\to\infty}{\simeq} 0$

i.e. $g_n(z) \xrightarrow[n\to\infty]{} g_\nu(z)$ a.s., where $g_\nu(z)$ is solution of

$$z g(z)^2 + z g(z) + 1 = 0$$

Note that there are a priori two solutions of this equation, but only one of them is a Stieltjes transform, and we already know that the functions g_n are Stieltjes transforms, so a continuity argument allows to conclude that:

$$g_\nu(z) = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{z}} \qquad \text{for } \operatorname{Im} z > 0$$

#

<u>NB</u>: we have skipped quite a lot of technical difficulties in this proof!