Random matrix theory: lecture 15

Marčenko-Pastur's theorem

Let $H$ be a $n \times n$ real or complex random matrix such that

(i) $\{h_{jk}, j,k=1\ldots n\}$ are iid random variables

(ii) $h_{jk}$ is a bounded random variable (i.e. $|h_{jk}(\omega)| \leq C \forall \omega$)

(iii) $E(h_{jk}) = 0$, $E(|h_{jk}|^2) = 1$

and let $W^{(n)} := \frac{1}{n} HH^*$, $\lambda^{(n)}_1 \ldots \lambda^{(n)}_n$ be the eigenvalues of $W^{(n)}$

(NB: $\lambda^{(n)}_i \geq 0$)

Remarks:

- assumption (ii) may be dropped without affecting the result.
- the assumption that $E(h_{jk}) = 0$ may also be dropped (see lecture 17) and one may also assume a variance different than 1; this changes slightly the limiting distribution.

Theorem (Marčenko-Pastur 1967, Bai-Silverstein 1995)

Under assumptions (i) - (iii),

$$F_n(t) := \frac{1}{n} \# \{j : \lambda^{(n)}_j \leq t\} \xrightarrow{n \to \infty} \int_0^t p_n(x) \, dx \text{ a.s. \ for } t \geq 0$$

where $p_n(x) = \frac{1}{\pi} \sqrt{\frac{4}{x} - \frac{4}{t}}$, $1 \leq x < t$

"quarter-circle" distribution
Remarks:

- Marčenko-Pastur's theorem is more general; see lecture 16.
- The "quarter-circle" distribution $p_\nu$ looks like:

\begin{align*}
\text{and is therefore not a quarter-circle...}
\end{align*}

But let us consider instead the limiting distribution of the singular values of $H^{(n)} := \frac{1}{\sqrt{n}} H$:

since these are the square roots of the $\lambda_j^{(n)}$,

their limiting distribution is:

$$q_\nu(x) = p_\nu(\sqrt{x^2}) \cdot 2x = \frac{1}{\nu} \sqrt{\nu - x^2} \text{ for } 0 \leq x \leq 2$$

which is indeed a quarter-circle:

Note that the singular values of $H^{(n)}$, which is non-hermitian, have a priori no relation with its eigenvalues. It can be shown however that

the limiting eigenvalue distribution of $H^{(n)}$ is the uniform distribution on the disc of radius 2 in the complex plane!
Proof (main ideas)

This time, we are going to use the Stieltjes transform.

\[ g_n(z) := \int_0^\infty \frac{1}{x-z} \, dF_n(x) = \frac{1}{n} \sum_{j=1}^n \frac{1}{y_j - z} = \frac{1}{n} \text{Tr} \left( (W^n - z I)^{-1} \right) \]

\[ g_\nu(z) := \int_0^\infty \frac{1}{x-z} \, P_\nu(x) \, dx = \frac{1}{\pi} \int_0^\infty \frac{1}{x-z} \, \sqrt{\frac{\pi}{2}} - \frac{1}{\pi} \, dx \]

\[ = \ldots = -\frac{1}{2} + \sqrt{\frac{z}{4} - \frac{1}{4}} \text{ for } \text{Im } z > 0 \]

NB: the reverse formula \( P_\nu(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \text{Im} (g_\nu(x + i\varepsilon)) \)

is actually easier to check!

More importantly, note that \( g_\nu(z) \) is solution of

the second order equation:

\[ z \cdot g(z)^2 + z \cdot g(z) + 1 = 0 \]

- We are now going to prove that:

\[ \lim_{n \to \infty} \sup_{z \in K} |g_n(z) - g_\nu(z)| = 0 \text{ a.s. } \forall K = [a, a] \times [b, b], a > 0, b > 0 \]

(we do not show uniform convergence here)

which implies the result.
First notice that \( W^{(n)} = \frac{1}{n^2} \sum_{k=1}^{n} h_k h_k^* \)
where \( h_k \) is the \( k \)th column of \( H \).

Let \( W^{(n)} = W^{(n)} - \frac{1}{n} h_k h_k^* = \frac{1}{n} \sum_{k=1}^{n} h_k h_k^* \).

and \( G^{(n)}(z) = (W^{(n)} - z I)^{-1} \), \( G_k^{(n)}(z) = (W_k^{(n)} - z I)^{-1} \).

Note that \( g_n(z) = \frac{1}{n^2} \text{Tr} G^{(n)}(z) \); we are interested in finding a limiting equation for \( g_n(z) \).

**Lemma 1**

\[
\frac{1}{n} h_k^* G^{(n)}(z) h_k = \frac{1}{n} \frac{h_k^* G_k^{(n)}(z) h_k}{1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k}
\]

**Proof**

\[
\left( h_k^* G_k^{(n)}(z) (W^{(n)} - z I) = h_k^* G_k^{(n)}(z) (W_k^{(n)} - z I + \frac{1}{n} h_k h_k^*) \right.
\]

\[
\left. = h_k^* + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k h_k^* = \left( 1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k \right) h_k^* \right)
\]

\[
\Rightarrow \quad h_k^* G_k^{(n)}(z) h_k = \left( 1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k \right) h_k^* G_k^{(n)}(z) h_k
\]

So \( h_k^* G_k^{(n)}(z) h_k = \left( 1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k \right) h_k^* G_k^{(n)}(z) h_k \).

**Lemma 2**

\[
g_n(z) = \frac{1}{n} \text{Tr} G^{(n)}(z) = -\frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} h_k^* G_k^{(n)}(z) h_k}
\]
Proof

\[ 1 = \frac{1}{n} \text{Tr}(I) = \frac{1}{n} \text{Tr}((W^{(m)} - zI) G^{(m)}(z)) \]

\[ = \frac{1}{n} \text{Tr} \left( \sum_{k=1}^{n} h_k^* h_k G^{(m)}(z) - 2G^{(m)}(z) \right) \]

\[ = \frac{1}{n} \sum_{k=1}^{n} h_k^* G^{(m)}(z) h_k - 2g_n(z) \]

\[ = \frac{1}{n} \sum_{k=1}^{n} \frac{h_k^* G^{(m)}(z) h_k}{1 + \frac{1}{n} h_k^* G^{(m)}(z) h_k} - 2g_n(z) \]

\[ \Rightarrow 2g_n(z) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\frac{1}{n} h_k^* G^{(m)}(z) h_k}{1 + \frac{1}{n} h_k^* G^{(m)}(z) h_k} - 1 \right) \]

\[ = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} h_k^* G^{(m)}(z) h_k} \]

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NB: This formula holds for any matrix of the form \( W^{(m)} = \frac{1}{n} HH^* \).

Lemma 3

\[ \frac{1}{n} h_k^* G_k^{(m)}(z) h_k \approx \frac{1}{n} \text{Tr}(G_k^{(m)}(z)) \text{ as } n \to \infty \]

more precisely:

\[ P \left( \left| \frac{1}{n} h_k^* G_k^{(m)}(z) h_k - \frac{1}{n} \text{Tr}(G_k^{(m)}(z)) \right| > \alpha \right) \leq \frac{C(\alpha)}{n^2} \]

which implies, by the Borel-Cantelli lemma,

that the difference converges to zero a.s. as \( n \to \infty \).
Proof idea

We only check here that expectations are equal:

\[ E \left( \frac{1}{n} \sum_{j \in k} G_k^{(n)}(z) \beta_k \right) = \frac{1}{n} \sum_{j \in k} E \left( \beta_j \left( W_k^{(n)} - 2I \right)^{-1}_{j:j} \beta_k \right) \]

Since \( W_k^{(n)} \) does not "contain" the k-th column of \( H \), it is independent of both \( \beta_j \) and \( \beta_k \) (assumption (ii)), so

\[ \ldots = \frac{1}{n} \sum_{j \neq k} \mathbb{E} \left( \beta_j \beta_k \right) \mathbb{E} \left( (W_k^{(n)} - 2I)^{-1}_{j:j} \right) = s_{j:k} \text{ by assumption (iii)} \]

\[ = \frac{1}{n} \sum_{j = 1}^{n} \mathbb{E} \left( (W_k^{(n)} - 2I)^{-1}_{j:j} \right) = \mathbb{E} \left( \frac{1}{n} \text{Tr} \ G_k^{(n)}(z) \right) \]

NB: assumption (ii) is needed in order to prove that a.s. convergence holds, but there is also a way to get rid of it.

Lemma 4

\[ \left| \frac{1}{n} \cdot \text{Tr} \ G_k^{(n)}(z) - \frac{1}{n} \cdot \text{Tr} \ G_k^{(n)}(z) \right| \leq \frac{1}{n} \cdot \text{Im} \ z^2 \quad \forall z \neq 0 \]

Proof: homework

NB: This lemma also holds for any matrix of the form \( W^{(n)} = \frac{1}{n} HH^* \).
Conclusion of the proof of the theorem:

By Lemmas 2, 3 & 4, we have:

\[
    g_m(z) = -\frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{n} \ln^* g_m^2(z) \ln^*}
\]

\[
    \sum_{n=0}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 + g_m(z)} = -\frac{1}{2} \frac{1}{1 + g_m(z)}
\]

So \( 2 \ g_m(z)^2 + 2 \ g_m(z) + 1 \xrightarrow[n \to \infty]{} 0 \)

i.e. \( g_m(z) \xrightarrow[n \to \infty]{} g'(z) \) as, where \( g'(z) \) is solution of

\[
    2g(z)^2 + 2g(z) + 1 = 0
\]

Note that there are a priori two solutions of this equation, but only one of them is a Stieltjes transform, and we already know that the functions \( g_m \) are Stieltjes transforms, so a continuity argument allows to conclude that:

\[
    g_m(z) = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2}} \quad \text{for } \text{Im } z > 0
\]

\#

NB: we have skipped quite a lot of technical difficulties in this proof!