Random matrix theory: lecture 1

1) General overview

The focus of the theory is on the statistics of eigenvalues of random matrices.

It has applications in a variety of fields:

- multivariate statistics
- nuclear theoretical physics
- wireless communications
- mathematical finance
- number theory (!) and more...

What are the typical questions of the theory?

Let H be a $n \times n$ matrix whose entries are random variables with a given joint distribution.

Reminder: the eigenvalues of H are the roots of the characteristic polynomial

$$C_H(\lambda) = \det(H - \lambda I)$$

Fact: for any H, C_H has n complex roots $\lambda_1 .. \lambda_n$

i.e. $C_H(\lambda) = c \prod_{j=1}^{n} (\lambda - \lambda_j)$ for some $c \in \mathbb{C}$

<u>NB</u>: since H is a random matrix, the eigenvalues

× $\lambda_1 .. \lambda_n$ are random variables

<u>Question 1</u>: what is the joint distribution $p(\lambda_1 .. \lambda_n)$
of the eigenvalues of H?

In general, it is very difficult to answer this question, but the answer is known for some specific
× ensembles of random matrices.

From now on (for this class), let us assume that H is real symmetric, so that the eigenvalues $\lambda_1 .. \lambda_n$ are real.

<u>Question 2</u>: what can be said about the
marginal distributions:

× main focus
$$\left\{\begin{array}{l} \cdot\ p(\lambda) = \int_{\mathbb{R}^{n-1}} p(\lambda, \lambda_2, .., \lambda_n)\, d\lambda_2 .. d\lambda_n \\ \cdot\ p(\lambda, \mu) = \int_{\mathbb{R}^{n-2}} p(\lambda, \mu, \lambda_3, .., \lambda_n)\, d\lambda_3 .. d\lambda_n \end{array}\right.$$

$\cdot\ p(\lambda, \mu, \nu) = \ldots$ etc. ?

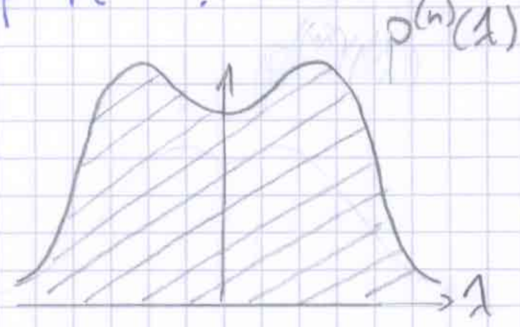
Again, the answer is known for specific ensembles only.

Let now $(H^{(n)})_{n\geq 1}$ be a sequence of $n\times n$ random matrices (therefore of increasing size).

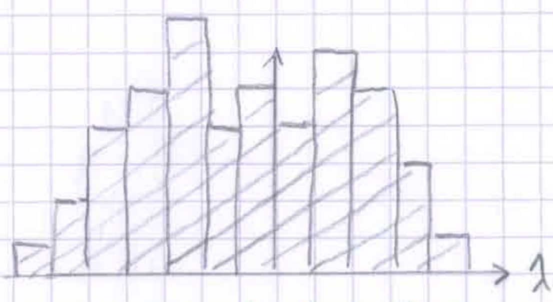
Let us assume that the corresponding marginal distributions $p^{(n)}(\lambda)$, $p^{(n)}(\lambda,\mu)$ are known for each n.

<u>Question 3</u>: what can be said about the limits $\lim_{n\to\infty} p^{(n)}(\lambda)$ & $\lim_{n\to\infty} p^{(n)}(\lambda,\mu)$?

Surprisingly, an answer to the first part of the question can be obtained for a much larger ensemble of random matrices, even if $p^{(n)}(\lambda)$ is not known for each n. This is done by looking at the empirical eigenvalue distribution of $H^{(n)}$:

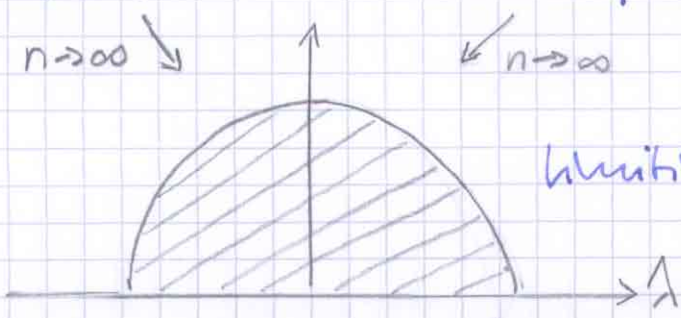


theoretical distribution at finite n



empirical distribution (= histogram) at finite n

$n\to\infty$ $\qquad$ $n\to\infty$



limiting distribution

2) History

1930-1950+ : multivariate statistics

John Wishart, 1928 : let $x^{(1)} \dots x^{(m)}$ be m independent samples of a n-variate Gaussian random vector
× $x \sim N_{\mathbb{R}}(0, I)$. Wishart computes the joint distribution of the entries of the $n \times n$ sample covariance matrix W defined as

× $$W_{jk} = \frac{1}{m} \sum_{\ell=1}^{m} x_j^{(\ell)} x_k^{(\ell)} \qquad \left[W = \frac{1}{m} X X^T \right]$$

Fisher, Girshik, Hsu, Roy, 1939 : compute the joint distribution of the eigenvalues of W.

James, 1964 : extension to the complex case

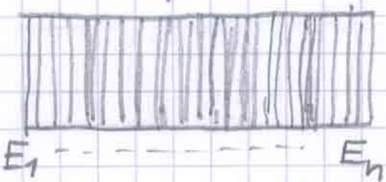
+ various extensions since then:

- joint eigenvalue distribution of $\frac{1}{m} X A X^*$, where A is a deterministic matrix
- distribution of the largest and smallest eigenvalues
- many more ...

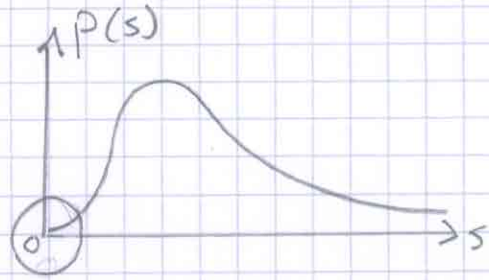
1950 - 1960+ : nuclear physics

Eugene Wigner, Freeman Dysan, Madan Lal Mehta

× Spectra of energy levels in heavy nuclei ($n \sim 235$):



$\Rightarrow$ statistics of spacings :



repulsion of energy levels

In quantum physics, energy levels are eigenvalues of a real symmetric matrix H, called the Hamiltonian.

With 235 particles or more inside a nucleus, it is hopeless to describe H exactly.

Wigner's idea: let us assume that H is completely random (but symmetric, still)!

Surprisingly, it works! (i.e. reproduces the above feature)

As a "by-product," Wigner obtains the limiting eigenvalue distribution of the studied matrices.

Many refinements and developments of Wigner's result have been obtained since and are still subject of active research.

<u>1960+</u>: <u>random matrix theory</u> (RMT)

The field expands rapidly and becomes a research subject on its own, mainly led by mathematical physicists at first, but also soon by pure mathematicians.

Some important dates:

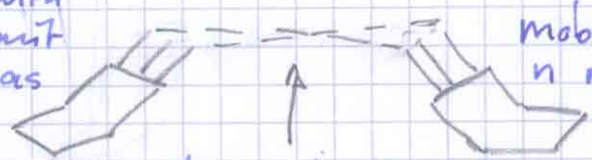
- 1967, Marčenko-Pastur:
  new technique for analyzing the limiting eigenvalue distribution of matrices of the form $A + XTX^*$, where X has i.i.d. entries
- 1991, Voiculescu: free probability and RMT
- 1994, Tracy-Widom: extreme eigenvalues
- 1995, Bogomolny & al: link with number theory

Finally, here are some names of active researchers in the field: Bai, Silverstein, Forrester, Baik, Ben Arous, Guionnet, Zeitouni, Soshnikov, Khorunzhy, Khoruzhenko, Goldsheid, Bohigas, Anderson, Its, Dembo, Diaconis and many others...

1990+ : applications to wireless communications

One example : multiple antenna systems

(Foschini - Gans, Telatar, 1995)



H is again too complicated to be described exactly $\Longrightarrow$ modelled as completely random

The capacity of such a system (i.e. the maximum number of bits that can be exchanged per second between the two mobile phones) is shown to be :

power constraint

$$
\begin{aligned}
C &= \mathbb{E}\left(\log \det\left(I + \frac{P}{n} HH^*\right)\right) \\
&= \mathbb{E}\left(\sum_{j=1}^{n} \log(1 + P\lambda_j)\right) \qquad \lambda_j = \text{e.v. of } \tfrac{1}{n} HH^* \\
&= \int_{\mathbb{R}_+^n} \sum_{j=1}^{n} \log(1+P\lambda_j)\, p(\lambda_1, .., \lambda_n)\, d\lambda_1 .. d\lambda_n \\
&= n \int_{\mathbb{R}_+} \log(1+P\lambda)\, p(\lambda)\, d\lambda
\end{aligned}
$$

Many other applications of RMT occur in wireless communications ...

3) Course plan

- Reviews: matrix analysis
  probability
- Finite-size analysis (n fixed)
- Asymptotic analysis ($n \to \infty$)
- Free probability

All along the way: applications to communications:

- Multiple antenna systems
- CDMA systems
- Ad hoc networks
- ISI channels

Projects to be defined on related topics