Statistical Ensembles of Complex, Quaternion, and Real Matrices

Jean Ginibre
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 7 July 1964)

Statistical ensembles of complex, quaternion, and real matrices with Gaussian probability distribution, are studied. We determine the over-all eigenvalue distribution in these three cases (in the real case, under the restriction that all eigenvalues are real). We also determine, in the complex case, all the correlation functions of the eigenvalues, as well as their limits when the order N of the matrices becomes infinite. In particular, the limit of the eigenvalue density as N → ∞ is constant over the whole complex plane.

INTRODUCTION

In order to obtain a theoretical description of highly excited regions of heavy-nuclei spectra, several authors\textsuperscript{1-4} have developed a theory of statistical matrix ensembles. The energy levels are the eigenvalues of the Hamiltonian H, which is regarded as an Hermitian matrix of very large order N. In the absence of any precise knowledge of H, one assumes a reasonable probability distribution for its matrix elements, from which one deduces statistical properties of its spectrum. An important simplification followed the introduction of unitary instead of Hermitian matrix ensembles.\textsuperscript{5} However, the physical origin of the problem—more precisely, the need to interpret part of the spectrum of a matrix as part of the energy spectrum of a physical system—has restricted the attention to matrix ensembles whose spectrum is contained in a one-dimensional line of the complex plane (the real axis for Hermitian, the unit circle for unitary matrices). In the present paper we study the eigenvalue distributions of matrix ensembles for which any point of the complex plane may belong to the spectrum. Apart from the intrinsic interest of the problem,\textsuperscript{6} one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications.

The definition of matrix ensembles consists of two parts:

1. The definition of an algebraic set of matrices Z. We shall consider: C, complex $N \times N$ matrices; Q, quaternion $N \times N$ matrices; R, real $N \times N$ matrices. The order is that of increasing difficulty. These are the Z's of Dyson's classification (Ref. 5, especially Theorem 7) and they can be characterized as the only irreducible matrix algebras over the real field (Refs. 5, 7 (especially Chap. 3)).

2. The choice of a measure or probability distribution on Z. Z is a finite-dimensional vector space over the real field. The linear measure $d\mu_{\mathcal{U}}$ which suggests itself naturally, is not suited for a probabilistic interpretation, for the measure of the whole space is infinite. We choose instead

$$d\mu(S) = d\mu_{\mathcal{U}}(S) \exp\left[-\left(1/4a^2\right) \text{Tr} S^I S \right]$$

(S generic element of Z, $a = \text{real positive constant}$) which satisfies the two properties:

(i) It is invariant under the adjoint representation of $Z_{\mathcal{U}}$, where $Z_{\mathcal{U}}$ is the group of unitary matrices in Z. More precisely, for any $U \in Z_{\mathcal{U}},$

$$d\mu(S) = d\mu(U^I SU)$$

(ii) The matrix elements of S are statistically independent.

These two properties are likely to determine $d\mu(S)$ uniquely.\textsuperscript{2} We study the three matrix ensembles thus obtained $Z_{c}, Z_{q},$ and $Z_{r}$ in Secs. 1, 2, and 3, respectively, with decreasing success. We obtain for $Z_{c}$ the over-all eigenvalue distribution and all the correlation functions of n eigenvalues ($1 \leq n \leq N$); for $Z_{q}$ the over-all eigenvalue distribution only; for $Z_{r}$ the over-all eigenvalue distribution in the restrictive case where all eigenvalues are real.

1. COMPLEX MATRICES

Z is the algebra of complex $N \times N$ matrices $S = (s_{ij}).$ The linear measure is defined by

$$d\mu_{\mathcal{U}}(S) = \prod_{i,j} dS_{ij} dS_{ij}^*$$

where \( dx \, dz^* \) means \( 2 \, dx \, dy \) if \( z = x + iy, z^* = x - iy \).
\( d\mu(S) \) is defined by \( (0 - 1) \).

From now on, we take \( 4a^2 = 1 \). We define the eigenvalue distribution \( P_N(z_1, \ldots, z_N) \) by

\[
\int d\mu(S) = P_N(z_1, \ldots, z_N) \prod_{i=1}^{N} dz_i \, dz_i^*.
\]  

(1.2)

where \( z_i \) (\( i = 1, \ldots, N \)) are the eigenvalues of the generic element \( S \in Z \).

\( \int \) means: for fixed \( (z_i) \), integration over all other variables. We now compute \( P_N \). Any \( S \in Z \) with distinct eigenvalues (we forget matrices with two equal eigenvalues, which form a set of measure 0) can be diagonalized

\[
S = XAX^{-1},
\]

(1.3)

where \( A \) is diagonal with \( A_{ii} = z_i \) and \( X \) is regular and is defined modulo multiplication on the right by any element of the commutator \( \alpha' \) of \( A \). \( \alpha' \) consists of all complex diagonal matrices. The correspondence \( S \rightarrow (A, X \mod \alpha') \) is \( 1 \rightarrow N! \) because there are \( N! \) ways to order the eigenvalues.

Infinitesimal variations \( dA, dX \) of \( A, X \) produce a variation \( dS \) of \( S \)

\[
dS = X(dx + [dR, A])X^{-1},
\]

(1.4)

where \( dR = X^{-1}dx \). From (1.4) and the invariance property: \( d\mu(S) = d\mu_1(XSX^{-1}) \) for any regular \( S \in Z \), we get

\[
d\mu_1(S) = \prod_{i \neq j} \frac{dz_i \, dz_j^*}{\prod_{i \neq j} [dR_i, A_{ij}] [dR_j, A_{jj}]},
\]

(1.5)

\[
d\mu_1(S) = \prod_{i < j} \frac{dz_i \, dz_j^* \prod_{i < k}}{\prod_{i < j} \prod_{i < k} dR_{ik} \, dR_{jk}^*}.
\]

(1.6)

We substitute (1.6) into (1.2) and get

\[
P_N(z_1, \ldots, z_N) = \frac{1}{N!} \prod_{i < j} (z_i - z_j)^4 J,
\]

(1.7)

where

\[
J = \int \prod_{i \neq j} dR_{ij} dR_{ij}^* \exp \left[-Tr \, \Sigma \, S \right].
\]

(1.8)

\( d\mu_0(X) = d\mu_0(U) \, d\mu_0(Y) \, d\mu_0(V) \),

(1.9)

is a unimodular subgroup \( \mathcal{G} \) of \( \mathbb{G} \). \( V \) is diagonal with real positive elements. The set of these matrices is also a subgroup \( \mathcal{V} \) of \( \mathbb{G} \).

(In other words, we have decomposed \( \mathbb{G} \) as the product of three subgroups \( Z_0, \mathcal{Y}, \mathcal{V} \). None of them is invariant and this decomposition has no simple algebraic property. However, it will provide a useful decomposition of \( d\mu_0(X) \).

**Proof:** Any set \( X^\alpha \) (\( \alpha = 1, \ldots, N \)) of \( N \) linearly independent vectors can be brought by a unitary change of basis into a form where the components \( X_i^\alpha \) satisfy \( X_i^\alpha = 0 \) for \( j > \alpha \). We apply this to the column vectors of \( X \) and obtain \( X = U_1 Y_1 \) with \( U_1 \) unitary and \( Y_1 \) triangular. \( Y_1 \) can then be written as \( Y_1 = U_0 YV \) with \( Y \) and \( V \) as above and \( U_0 \) diagonal and unitary. Therefore \( X = UYV \) with \( U = U_1 U_0 \). If now \( X = UV \), \( \alpha \) is unitary (left-hand side) and triangular with real positive diagonal elements (right-hand side); therefore \( B = 1 \), and the decomposition is unique.

We now change the variables in \( d\mu_0(X) \) from \( dR \) to \( dU, dY, dV \). We start from

\[
dR = X^{-1} dX = V^{-1} dV + V^{-1} Y^{-1} dY \, dV + V^{-1} Y^{-1} U^{-1} dU \, dY \, dV.
\]

(1.10)

The first term contributes to the real part of the diagonal elements; The second term, to the \( dR_{ii} \) with \( i < j \). In the third term, \( dL = -i U^{-1} dU \) is Hermitian. From \( Y \in \mathcal{Y}, Y^{-1} \in \mathcal{Y} \) it follows that, for \( i \geq j \),

\[
(Y^{-1} dL \, Y)_{ii} = dL_{ii} + \text{linear combination of } dL_{ki}
\]

with \( k - l > i - j \).

(1.11)

It follows from these remarks that

\[
d\mu_0(X) = d\mu_0(U) \, d\mu_0(Y) \, d\mu_0(V),
\]

(1.12)

where

\[
d\mu_0(U) = \prod_{i < j} dL_{ij} \prod_{i} dL_{ii} = \prod_{i} dL_{ii},
\]

(1.13)

\[
d\mu_0(Y) = \prod_{i < j} (Y^{-1} dY)_{ij} (Y^{-1} dY)_{ji},
\]

(1.14)

\[
d\mu_0(V) = \prod_{i} (2 V^{-1} \, dV_{ii}),
\]

(1.15)

are the invariant measures on \( Z_0, \mathcal{Y}, \mathcal{V} \) respectively. We need the quotient measure on \( \mathbb{G}/\alpha' \),

\[
\prod_{i \neq j} dR_{ij} dR_{ij}^* = d\mu_0(\mathcal{G}) / d\mu_0(\alpha').
\]

(1.16)

\( \alpha' \) is the direct product of \( \mathcal{V} \) and the group \( \mathcal{U}_0 \) of unitary diagonal matrices, with

\[
d\mu_0(\alpha') = d\mu_0(\mathcal{U}_0) \, d\mu_0(\mathcal{V}).
\]

(1.17)
We now come back to (1.8). From (1.3)
\[ \text{Tr } S^tS = \text{Tr } A^tHAH^{-1}, \tag{1.18} \]
where $H$ should be $X^tX$. But $V$ commutes with $A$ and therefore disappears, as expected, which enables us to take $H = Y^tY$. We then get from (1.8), (1.12), (1.16)
\[ J \int d\mu_0(\mathcal U_0) = \int \exp[-\text{Tr } S^tS] \, d\mu_0(U) \, d\mu_0(Y) \tag{1.19} \]
or
\[ J = \frac{\Omega_U}{(2\pi)^N} \int \exp[-\text{Tr } S^tS] \, d\mu_0(Y), \tag{1.20} \]
where $(2\pi)^N$ is the volume of $\mathcal U_0$ and $\Omega_U = \int d\mu_0(U)$ is the volume of $Z_0$. We next perform a last change of variables from $Y$ to $H$. Any $n \times n$ upper left block $Y_n$ of any $Y \in \mathcal Y$ has determinant 1. This implies that the jacobian of the transformation from $(dY)_{ij}$ to $(Y^{-1}dY)_{ij}$ is 1. Therefore
\[ d\mu_0(Y) = \prod_{i<j} dY_{ij} \prod_{i} dY_{ii}. \tag{1.21} \]
$H$ is defined by $H = Y^tY$ or more explicitly
\[ \begin{cases} H_{ij} = Y_{ij} + \sum_{k<i} Y_{ik} Y_{kj} & \text{for } i < j, \\ H_{ii} = Y_{ii}^* + \sum_{k<i} Y_{ik}^* Y_{kj} & \text{for } i > j. \end{cases} \tag{1.22} \]
Each $n \times n$ upper left block $H_n$ of $H$ can be defined by $H_n = Y_n^* Y_n$. Therefore $\det H_n = 1$ determines the diagonal elements of $H_n$ by the condition $\det H_n = 1$. Each diagonal element of $H$ is then defined (by induction) as a polynomial with integer coefficients in the nondiagonal elements.

It follows immediately from (1.22) that the correspondence $Y \to H$ is one to one and that
\[ d\mu_0(Y) = \prod_{i<j} dH_{ij} \prod_{i} dH_{ii} = \prod_{i<j} dH_{ij}. \tag{1.23} \]
We now perform the integration over $H$ in $N$ steps; each one consists of integrating over the variables of the last row and column of $H$, and brings back to the original problem for matrices of order smaller by one unit, obtained from the original ones by removing the last row and column. The proof rests on a recursion formula which we now derive:

Let $H', A'$, etc. be the relevant matrices of order $n; H, A$ the matrices of order $n - 1$, more precisely the $(n - 1) \times (n - 1)$ upper left blocks of $H', A'$. Greek (Latin) indices run from 1 to $n (n - 1)$. From $\det H' = 1$, it follows that $H'_{\alpha \beta}^{-1} = \Delta_{\beta \alpha}$, where $\Delta_{\alpha \beta}$ is the minor $(\alpha, \beta)$ of $H'$. Similarly $H_{ii}^{-1} = \Delta_{ii}$, where $\Delta_{ii}$ is the minor $(ii)$ of $H$. Now let
\[ \phi_n = \text{Tr } S'^tS' = \text{Tr } A'H' A'H'^{-1} \]
\[ = \sum_{\alpha, \beta} H'_{\alpha \beta} \Delta_{\beta \alpha}. \tag{1.24} \]
We separate the last row and column; let $e = (e_i), i = 1, \ldots, n - 1$, where $e_i = H'_{i\alpha}$. Then
\[ \phi_n = |e|^2 H'_{nn} - \sum_{i,k} e_i e_k \Delta_{ik} - \sum_{i,k} e_i^2 e_k \Delta_{ii} + \sum_{i,j,k,l} H'_{ii} \Delta_{jk} (\Delta_{kj} - e_i e_j e_k e_l). \tag{1.25} \]
where $\Delta_{jk}$ is the minor of $H$ obtained by removing the rows $i, k$ and the columns $j, l$. From $\det H' = \det H = 1$, we get
\[ H'_{nn} = 1 + \sum_{k,l} e_k e_l \Delta_{kl}. \tag{1.26} \]
Substituting in (1.26) and using the elementary identity
\[ \Delta_{ii} = \Delta_{ii}; \tag{1.27} \]
we get, after straightforward algebra,
\[ \phi_n = |e|^2 \phi_{n-1} + \langle e^* | H^{-1}(A^* - z_0^*) H(A - z_0) H^{-1} | e \rangle, \tag{1.28} \]
where
\[ \langle e^* | B \, | e \rangle = \sum_{k,l} e_k^* B_{kl} e_l. \tag{1.29} \]
We now substitute (1.28) for $n = N$ into (1.20) and get
\[ J = \frac{\Omega_U}{(2\pi)^N} \int \exp[-|z|^2 - \phi_{N-1}] \]
\[ \times \prod_{i<j,1 \leq i < j \leq N-1} dH_{ij} \prod_{i} dH_{ii} \int \prod_{i=1}^{N-1} \exp \left[-\langle e^* | H^{-1}(A^* - z_0^*) H(A - z_0) H^{-1} | e \rangle \right]. \tag{1.30} \]
The last integration is straightforward and gives
\[ (2\pi)^{-N} \left[ \prod_{i=1}^{N-1} |z_i - z_N|^2 \right]^{-1}. \]
The same procedure can be repeated $N$ times and gives
\[ J = \Omega_U (2\pi)^{(N-3)/2} \prod_{i<j} |z_i - z_j|^2 \int \prod_{i} \exp \left[-\sum_{i=1}^{N} |z_i|^2 \right]. \tag{1.31} \]
$\Omega_U$ is easily computed. With the normalization

\[ \Omega_U \text{ is easily computed.} \]

With the normalization
(1.13), we get
\[ \Omega_U = (2\pi)^{(N+n+1)/2}/N! \cdot (N-1)!. \]  (1.32)

(1.31), (1.32), and (1.7) then give
\[
P_N(z_1, \ldots, z_N) = \frac{(2\pi)^{N(N-1)}}{N!} \prod_{i \neq j} |z_i - z_j|^2 \times \exp \left[ -\sum |z_i|^2 \right]. \]  (1.33)
P_N is normalized according to
\[
\int P_N(z_1, \ldots, z_N) \prod dz_i \cdot \text{d}z_i^* \, \text{d}z_i^* \nonumber
f \rightarrow \infty \quad \text{and} \quad \text{as} \nonumber
\]
Choose a set of \( N - n \) exponents \( p'_i \), i.e., \( N - n \) integers between 0 and \( N - 1 \), which are to be distributed among the integration variables \( z_{n+1}, \ldots, z_N \).

Distribute them among these variables, which gives a factor \( (N-n)! \) and cancels the corresponding factor in (1.37).

Integrate over the \( N - n \) angles \( \theta_i (i = n+1, \ldots, N) \), which gives a factor \((2\pi)^{N-n}\).

Integrate over the \( N - n \) variables \( |z_i| = r_i \), which gives one factor \( p'_i \) for each \( p'_i \) and cancels the corresponding factor in \( P_N \).

Distribute the remaining set of exponents \( p_i \)
ential series give
\[ 1 - 2\pi \rho_N(z) \leq \exp \left( -r^2 \right) \frac{N}{N!} \frac{1 + N - r^2}{1 + N} \]
for \( r^2 \leq N \), \( (1.44) \)
\[ 2\pi \rho_N(z) \leq \exp \left( -r^2 \right) \frac{N}{N!} \frac{1 + N - r^2}{1 + N} \]
for \( r^2 \geq N \). \( (1.45) \)

For \( r = N^{1/2} \leq u \leq 1 \ll N \), the leading term in the right-hand side of \((1.44)\) and \((1.45)\) is \( \exp (-2u^2)/2u(2\pi)^{1/2} \). This gives a sharp fall of \( \rho_N(z) \) from \( \rho \) to 0 when \( r \) varies in an interval of order 1 around \( N^{1/2} \). As a consequence, the number of eigenvalues in the "tail" of the distribution, defined as
\[ \delta N = \int_{|z| > N^{1/2}} \rho_N(z) \, dz \, dz^* \]
(1.46)
is proportional to \( N^{1/2} \). More precisely \( \delta N \simeq (N/2\pi)^{1/2} \) for \( N \gg 1 \).

**Electrostatic Analogy**

The electrostatic model introduced by Wigner\(^8\) and Dyson\(^7\) can be extended to the present case. Consider \( N \) unit charges in a two-dimensional space, which is taken as the complex plane of the variable \( z \). The positions of the charges are \( z_1, \ldots, z_N \). Suppose that the charges move in an harmonic oscillator potential \( \frac{1}{2} |z|^2 \) centered at the origin. Then the potential energy of the system is
\[ U(z_1, \ldots, z_N) = -\sum_{i<j} \log |z_i - z_j| + \frac{1}{2} \sum_i |z_i|^2. \]
(1.47)
The probability distribution of the positions \( z_1, \ldots, z_N \) when this Coulomb gas is in thermodynamical equilibrium at the temperature \( T \) is proportional to \( \exp \left[ -\beta U(z_1, \ldots, z_N) \right] \) (where \( \beta = 1/kT \)). For \( \beta = 2 \) this is proportional to \( P_N \).

Therefore the distribution of eigenvalues of a random matrix \( S \in Z_0 \) is identical with the distribution of the positions of charges of a two-dimensional Coulomb gas in a harmonic oscillator potential, at a temperature corresponding to \( \beta = 2 \).

2. QUATERNION MATRICES

\( Z \) is now the algebra \( Z_0 \) of \( N \times N \) \( Q \) matrices, i.e., matrices with coefficients in the quaternion field \( Q \). \( Q \) can be represented as a two-dimensional complex vector space.\(^9\) This representation associates to any \( N \times N \) \( Q \) matrix a \( 2N \times 2N \) complex matrix. The image of matrices in \( Z_0 \) are characterized by
\[ i T = T^*, \]
where \( * \) means complex conjugate and \( T \) is a direct sum of \( N \) \( 2 \times 2 \) blocks of the form \( \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \). (These matrices are quaternion real in Dyson's notations.\(^4\))

The group \( Z_0 \) of unitary matrices in \( Z \) is characterized by \((2.1)\) and \( T^T T = 1 \), and satisfies therefore also \( T^T i T = i \), where \( \tau \) means transposed. \( Z_0 \) is the symplectic group \( Sp(N) \). Let \( S \in Z_0 \). We first look for eigenvalues of \( S \) in \( Q \), i.e., for \( \lambda \in Q \) and vectors \( v \in Q^N \) such that \( S v = \lambda v \). (The product of a vector \( \in Q^N \) by a scalar \( \lambda \) is the product by \( \lambda \) on the right.) Now,

**Lemma:** If \( \lambda \in Q \) is an eigenvalue of \( S \), then \( \lambda^{-1} \mu \) is also an eigenvalue of \( S \) for any \( \mu \neq 0 \), \( \mu \in Q \). In fact,
\[ S v = \lambda v \] implies \( S(\mu v) = \lambda \mu^{-1} \mu \).
(2.2)

Therefore the eigenvalues of \( S \) in \( Q \) constitute orbits, each of which is the set of quaternions obtained from one \( \lambda \in Q \) by the internal automorphisms \( \lambda \rightarrow \mu^{-1} \lambda \mu \) (which are simply three-dimensional rotations of the imaginary part).

Consider now a subspace of \( Q \) isomorphic to \( C \). This subspace intersects every orbit in two points which are complex conjugate in \( C \) (this means simply that in the three-dimensional space of purely imaginary quaternions, we consider the intersection of a sphere with one of its diameters), and these two points determine the orbit completely. Therefore all possible information on the eigenvalues of a \( Q \) matrix in \( Q \) can be obtained from the eigenvalues in a subspace \( C \) of \( Q \).

From now on we use only the \( 2N \times 2N \) complex representation of the matrices of \( Z_0 \), and study their (complex) eigenvalues in the ordinary sense. One verifies directly the above mentioned result, namely that due to \((2.1)\), the eigenvalues of any \( S \in Z_0 \) are \( 2 \times 2 \) complex conjugate. For if \( S v = \lambda v \), \( v \in C^N \), \( v \in C^N \), then \( \bar{\lambda} \bar{v} = \lambda \bar{v} \), or \( S(\bar{\lambda} \bar{v}) = \lambda^{-1} \lambda \bar{v} \).

Moreover, the eigenvector associated with \( \lambda^{-1} \lambda \). \( v \) can be taken to be \( v \bar{v}^{-1} \). We shall need the following

**Lemma:** If \( S \in Z_0 \), and if all eigenvalues of \( S \) are distinct, then \( S = XX^* \), where \( X, A \in Z_0 \), and \( A \) is diagonal.

In fact, \( S \) can be diagonalized as a complex matrix. One can take \( A \) as a direct sum of \( 2 \times 2 \)


blocks
\[
\begin{pmatrix}
z_i & 0 \\
0 & z_i^* 
\end{pmatrix}, \quad (i = 1, \ldots, N).
\]
Therefore \(A \in Z_q\). The column vectors of \(X\) are the eigenvectors \(X^a\) of \(S\). One can take \(X^a = \gamma(X^a)^{\ast -1}\) \((n = 1, \ldots, N)\), which is equivalent to \(X \in Z_q\).

We now define measures on \(Z_q\). The linear measure is
\[
d\mu_L(S) = \prod_{i,j} dS_{ij}. \tag{2.3}
\]
[Note that because of (2.1), the factors in (2.3) are \(2 \times 2\) complex conjugate.]
\[
d\mu(S) = d\mu_L(S) \exp \left[-\frac{1}{2} \text{Tr} S^T S \right]. \tag{2.4}
\]
The coefficient \(\frac{1}{2}\) is intended to achieve greater similarity with the previous case and to compensate for the artificial doubling of the dimension of the matrices.

\(P_N\) is defined by (1.2). The first step in its calculation is the same as previously and leads to
\[
d\mu_L(S) = \prod_{i=1}^N |z_i - z_i^*|^2 \prod_{i<j} \left| z_i - z_j \right| \left| z_i - z_j^* \right| 
\times \prod_{i=1}^N dz_i dz_i^* \prod_{i<j} dR_{ij} \tag{2.5}
\]
with \(dR = X^{-1} dX\), and therefore to
\[
P_N(z_1, \ldots, z_N) = \left( \frac{1}{N!2^n} \right) \prod_{i=1}^N \left| z_i - z_i^* \right|^2 
\times \prod_{i<j} \left| z_i - z_j \right| \left| z_i - z_j^* \right| J, \tag{2.6}
\]
where now
\[
J = \int \prod_{i<j} dR_{ij} \exp \left[-\frac{1}{2} \text{Tr} S^T S \right]. \tag{2.7}
\]
The factor \(2^N\) in (2.6) comes from the exchanges \(z_i \leftrightarrow z_i^*\). \(d\mu_0(X) = \prod_{i,j} dR_{ij}\) is the invariant measure on the group \(G\) of regular elements in \(Z_q\) and \(\prod_{i,j} dR_{ij}\) is the quotient measure on \(G/\alpha'\) where \(\alpha'\) is the commutator of \(A\) in \(Z_q\). Any \(X \in G\) can be decomposed as previously as a product \(X = UV\) where \(U\) is unitary, \(Y\) is triangular \((Y_{ij} = 0 \text{ for } i > j)\) and satisfies \(Y_{ii} = 1\), and \(V\) is diagonal with real positive elements. Moreover, it follows from \(\gamma X = X^\ast \gamma\) that
\[
B = Y^* V^T V^{-1} Y^{-1} = U^{*\ast} \gamma U \tag{2.8}
\]
is unitary and antisymmetric (right-hand side), and has nonzero elements only in \(2 \times 2\) blocks along the diagonal (comparison of both sides). From this and the condition \(V_{ii} > 0\), it follows easily that \(B = \gamma\). Therefore \(U \in Z_{q}(\subset Z_q)\). It then follows that \(Y\) and \(V\) belong to \(Z_q\). In particular,
\[
V_{2i-1,2i-1} = V_{2i,2i}
\]
and
\[
Y_{2i-1,2i} = Y_{2i-1,2i} = 0 \quad (i = 1, \ldots, N).
\]
The same argument as previously then leads to
\[
J = \frac{\Omega_\gamma}{(2\pi)^N} \int \exp \left[-\frac{1}{2} \text{Tr} S^T S \right] d\mu_0(Y), \tag{2.9}
\]
where \(\Omega_\gamma\) is the volume of the symplectic group and is defined by
\[
\Omega_\gamma = \int_{S_n} d\mu_0(U) = \int \prod_{i,j} (iU^T i U_{ij}) \tag{2.10}
\]
and
\[
d\mu_0(Y) = \prod (Y^{-1} dY)_{ii} = \prod dY_{ii}, \tag{2.11}
\]
where the product extends over the elements \(i < j\), \((i,j) \neq (2k-1,2k)\) for \(k = 1, \ldots, N\). We next change the integration variables from \(Y\) to \(H = Y' Y\). Then
\[
d\mu_0(Y) = \prod dH_{ii}, \tag{2.12}
\]
where \(\prod\) has the same meaning as in (2.11). We integrate over \(H\) in \(N\) steps as previously. We first derive a recursion formula analogous to (1.28). \(H', A', \ldots\) now denote matrices of order \(2n, \Delta'\) are the minors of \(H'\). \(H, A\) are the \((2n - 1) \times 2(n - 1)\) upper left blocks of \(H', A', \Delta\) are the minors of \(H\). From \(H' \in Z_q\) and \(H' = Y' Y'\), it follows that the \(2 \times 2\) diagonal blocks of \(H'\) have the form
\[
\begin{pmatrix}
h_i & 0 \\
0 & h_i
\end{pmatrix}
\]
In particular,
\[
H_{2n-1,2n-1} = H_{2n,2n} = h_n.
\]
Let
\[
e = (e_i), \quad e' = (e'_i) \quad [i = 1, \ldots, 2(n - 1)]
\]
where \(e_i = H_{i,2n-1}\) and \(e'_i = H'_{i,2n}\). Then
\[
e' = \gamma e'. \tag{2.13}
\]
The diagonal elements of \(H'\) are defined by the condition that all upper left blocks of \(H'\) have determinant one. In particular,
\[
h_n = 1 + \sum_{i,j} e'_i e_j \Delta_{ij}, \tag{2.14}
\]
\[
h_n = 1 + \sum_{k,l} e_k e_l \Delta_{kl} - \sum_{i} e'_i \Delta_{ii}. \tag{2.15}
\]
(2.14) defines $h_*$ and (2.14), (2.15) give an identity between $e$, $e'$, and $\Delta$.

$$
\sum_{i,j} \langle e^* e_i - e_i^* e_j \rangle \Delta_{ij} = \sum_{i,j} e_i^* e_j e_i^* e_j (\Delta_{ij} - \Delta_{ji}).
$$

(2.16)

Now from (2.13) and $H^{-1} \in \mathbb{Z}_Q$, it follows easily that

$$
\langle e^* | H^{-1} | e' \rangle = \langle e^* | H^{-1} | e \rangle,
$$

(2.17)

$$
\langle e^* | H^{-1} | e \rangle = -\langle (e^* | H^{-1} | e')^* \rangle.
$$

(2.18)

The notation is that of (1.29). From (2.17), (2.18), and $H_i^{-1} = \Delta_{ii}$, it follows that both sides of (2.16) are zero, and (2.16) reduces to

$$
\langle e^* | H^{-1} | e \rangle = \langle e^* | H^{-1} | e' \rangle = 0.
$$

(2.19)

Let now

$$
\phi_* = \text{Tr} A^* H A H^{-1}.
$$

We separate the last two rows and columns of the various matrices by applying (1.28) twice.

$$
\phi_* = \phi_{n-1} + 2 \langle e^* \rangle U \langle e \rangle + \langle e^* \rangle V \langle e' \rangle,
$$

(2.20)

where

$$
U = H^{-1} (A^* - z_n^2) H (A - z_n^2) H^{-1}
$$

(2.21)

and

$$
V_{ii} = \sum [z_i^2 - z_i z_i^* - z_i z_i^* e_{e_i}^* (\Delta_{ii} - \Delta_{ii}^*)]
$$

$$
- \sum e_{e_i}^* (h_{ii} - e_{e_i}^* e_{e_i}) \Delta_{ii}^* e_{e_i}
$$

$$
+ (\Delta_{ii}^* e_{e_i} + z_i e_{e_i}^* e_{e_i} (\Delta_{ii} - \Delta_{ii}^* e_{e_i}))
$$

(2.22)

(summation over all indices $i$, $j$, $i$, $j$).

In (2.22) some terms have cancelled out because of (2.16), (2.19), and the $z_i^*$ for $i = 1, \ldots, 2(n - 1)$ are the $(z_1, z_1^*, z_2, z_2^*, \ldots, z_{n-1}, z_{n-1}^*)$ in that order. We next substitute (2.14) into (2.22). After some straightforward algebra involving repeated use of (2.13), (2.17), (2.19), (1.27) and its analogue for

$$
\Delta_{ii}^* e_{e_i}
$$

the terms of fourth order in $e$, $e'$ cancel out and we get

$$
\phi_* = \phi_{n-1} + 2 \langle z_1^2 \rangle + 2 \langle e^* \rangle U \langle e \rangle.
$$

(2.23)

The integration over $z$ runs exactly as in the complex case and gives after $N$ successive steps

$$
J = \Omega_N (2\pi)^{N(N-1)/2} \left[ \prod_{i < j \leq N} |z_i - z_j|^2 |z_i - z_j^*|^2 \right]^{-1}
$$

$$
\times \exp \left[ - \sum_{i=1}^N |z_i|^2 \right].
$$

(2.24)

(2.25)

(2.26)

$P_N$ is normalized according to

$$
\int_{\mathbb{C}^N} P_N (z_1, \ldots, z_N) \prod dz_i dz_i^* = (2\pi)^{2N},
$$

(2.27)

which follows from (2.4) and (1.2) by direct computation. By analogy with (1.35), we define

$$
P_N (z_1, \ldots, z_N) = \left[ (2\pi)^{N(11)} \prod (2N - 1) \right]^{-1}
$$

$$
\times \exp \left[ - \sum_{i=1}^N |z_i|^2 \right]
$$

$$
\times \prod_{i < j \leq N} |z_i - z_j|^2 |z_i - z_j^*|^2,
$$

(2.28)

which is normalized according to (1.36).

The determination of the correlation functions appears to be considerably more difficult than in the complex case, and the electrostatic interpretation of $P_N$ breaks down.

### 3. REAL MATRICES

$Z$ is now the algebra of real $N \times N$ matrices $S = (S_{ij})$. The linear measure is

$$
d\mu_S (S) = \prod_{i,j} dS_{ij}.
$$

(3.1)

The eigenvalues of $S$ now consist of $\nu$ complex-conjugate pairs $(0 \leq 2\nu \leq N)$ and $q = N - 2\nu$ real numbers: $z_{2i} = z_{2i+1}$ for $i = 1, \ldots, \nu$, and $z_i$ for $i > 2\nu$.

We define $P_N (z_1, \ldots, z_N)$ by

$$
\int d\mu (S) = P_N (z_1, \ldots, z_N) \prod_{i=1}^N dz_i
$$

(3.2)

where $\mu$ has the same meaning as in (1.2).

Any $S$ with distinct eigenvalues can be diagonalized: $S = XAX^{-1}$. $A$ is diagonal with $A_{ii} = z_i$. $X$ is regular. Its first $2\nu$ column vectors $X^*$ are $2 \times 2$ complex conjugate and the last $q$ are real:

$$
X^2i = (X^{2i-1})^* \quad \text{for} \quad 1 \leq i \leq \nu,
$$

and

$$
X^j = X^j* \quad \text{for} \quad j > 2\nu.
$$

(3.3)
$X$ is defined modulo multiplication on the right by any element of the group $\mathcal{W}$ of diagonal matrices $W$ which satisfy:

$$W_{j+1,j} = W_{j+1,j-1}^* \quad \text{for} \quad 1 \leq j \leq \nu$$

and

$$W_{j,j} = W_{j,j}^* \quad \text{for} \quad j > 2\nu.$$ 

The correspondence $S \rightarrow (A, X \mod \mathcal{W})$ is $1 \rightarrow \nu! (N - 2\nu)! 2^\nu$. From (1.4) we get

$$d\mu_\nu(S) = \prod_i d\xi_i \prod_{i \neq j} |z_i - z_j|^2 \prod_{i,j} dR_{i,j},$$

where $dR = X^{-1}dX$ or

$$d\mu_\nu(S) = \prod_i d\xi_i \prod_{i \neq j} |z_i - z_j|^2 \prod_{i,j} dR_{i,j}. $$

Therefore

$$P_\nu(z_1, \cdots, z_N) = [\nu!(N - 2\nu)!2^\nu]^{-1} \prod_{i \neq j} |z_i - z_j|^2 J,$$

where

$$J = \int \prod_{i,j} dR_{i,j} \exp[-\text{Tr} S^T S].$$

We define a measure on the set $\mathcal{X}$ of the $X$ which satisfy (3.3) by

$$d\mu_0(X) = \prod_{i,j} dR_{i,j};$$

$\mathcal{X}$ is not a group. However the relation: $X_1 \sim X_2$ if $X_1^{-1}X_2 \in \mathcal{W}$ is an equivalence relation in $\mathcal{X}$. We pick one element $X_0$ in each class. Then any $X \in \mathcal{X}$ can be written as $X = X_0W$ in one and only one way. Then

$$dR = X^{-1}dX = W^{-1}dW + W^{-1}X_0^{-1}dX_0W$$

(3.9) implies

$$d\mu_0(X) = d\mu_0(W) \prod_{i,j} dR_{i,j},$$

where

$$d\mu_0(W) = \prod_{i,j} (W^{-1}dW)_{i,j}$$

is the invariant measure on $\mathcal{W}$. $\mathcal{W}$ is the direct product of the subgroup $\mathcal{U}_0 \subset \mathcal{W}$ of matrices $U_0$ which satisfy

$$(U_0)_{j+1,j-1} = (U_0)_{j+1,j-1}^* = e^{i\theta} \quad \text{for} \quad 1 \leq j \leq \nu$$

and

$$(U_0)_{j,j} = \pm 1 \quad \text{for} \quad j > 2\nu$$

and the subgroup $\mathfrak{U}$ of real positive matrices $V \in \mathcal{W}$:

$$V_{j+1,j} = V_{j+1,j-1} \quad \text{for} \quad 1 \leq j \leq \nu,$$

$$V_{j,j} > 0 \quad \text{for} \quad 1 \leq j \leq N.$$ 

Furthermore,

$$d\mu_0(V) = d\mu_0(U_0) \mu_0(U),$$

where

$$d\mu_0(U_0) = \prod_{i=1}^\nu d\theta_i,$$

$$d\mu_0(V) = \prod_{i,j=1}^\nu 2V_{i,j}^{-1} dV_{i,j} = \prod_{i,j=1}^\nu V_{i,j}^{-1} dV_{i,j}$$

are the invariant measures on $\mathcal{U}_0$ and $\mathfrak{U}$.

Note also that the sum over the $\pm 1$ of the $N - 2\nu$ last diagonal elements of $U_0$ will result in a factor $2^{n-2\nu}$ in the volume of this group.

We next introduce a representation where only real matrices appear. Let $\xi$ be the unitary matrix

$$\xi = \bigoplus_{i} \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) \otimes I_{N-2\nu},$$

where $\bigoplus$ means direct sum. Then $X' = X\xi$ is real. Furthermore

$$d\mu_0(X') = \prod_{i,j} (X'^{-1}dX')_{i,j} = d\mu_0(X).$$

(3.15)

$X'$ can be written in one and only one way as $X' = OYV'$ where $O$, $Y$, $V'$ are real, $O$ is orthogonal (proper or not), $Y$ is triangular (i.e., $Y_{jj} = 0$ for $i > j$) with $Y_{ii} = 1$, and $V'$ is real $> 0$ and diagonal. Then it follows from

$$X'^{-1}dX' = V'^{-1}dV' + V'^{-1}Y^{-1}dY V'$$

$$+ V'^{-1}Y^{-1}(O^{-1}dO) Y' V'$$

(3.16)

and from (3.15) that

$$d\mu_0(X') = d\mu_0(O) d\mu_0(Y) d\mu_0(V')$$

(3.17)

where

$$d\mu_0(O) = \prod_{i,j} (O^{-1}dO)_{i,j},$$

$$d\mu_0(Y) = \prod_{i,j} (Y^{-1}dY)_{i,j} = \prod_{i,j} dY_{i,j},$$

$$d\mu_0(V') = \prod_{i,j} V'^{-1}dV_{i,j}.$$ 

(3.20)

$V'$ can be decomposed in one and only one way as $V' = VT$ where $V \in \mathfrak{U}$ and

$$T = \bigoplus_{i} \left( \begin{array}{cc} t_i & 0 \\ 0 & 1/t_i \end{array} \right) \otimes I_{N-2\nu}$$

(3.21)

with

$$d\mu_0(V') = d\mu_0(V) d\mu_0(T),$$

(3.22)
where \( d\mu_0 (V) \) is defined in (3.13) and
\[
d\mu_0 (T) = \prod_{i=1}^{2j - 1} dt_i / t_i. \tag{3.22}
\]

We now come back to (3.7). Now \( \text{Tr} \ S' S = \text{Tr} \ A' H A H^{-1} \), where \( H \) should be \( X' X = \xi X' X' \). However, due to \( X' = OYVT \) and the fact that \( V \) commutes with both \( \xi \) and \( A \), one can take \( H = \xi H' \), where
\[
H' = T Y' Y T \tag{3.23}
\]
depends only on \( Y \) and \( T \). We now introduce in (3.7) an extra integration over \( u \), and make use of (3.10), (3.11), (3.17), and (3.22). We obtain
\[
J = \frac{\Omega_0}{(2\pi)^{2^{N-2}\nu}} \int \exp \left[ -\text{Tr} \ S' S \right] d\mu_0 (Y) d\mu_0 (T), \tag{3.24}
\]
where \( \Omega_0 \) is the volume of the full orthogonal group and \( (2\pi)^{2^{N-2}\nu} \) is the volume of \( u_0 \). Now it follows from (3.23) that
\[
d\mu_0 (Y) = \prod_{i<j} dY_{ij} = \prod_{i<j} d(Y' Y)_{ij} = \prod_{i<j} dH_{ij}, \tag{3.25}
\]
the Jacobian of the last transformation being \((\det T)^{N-1} = 1\). Finally,
\[
J = \frac{\Omega_0}{(2\pi)^{2^{N-2}\nu}} \int \exp \left[ -\text{Tr} \ S' S \right] \prod_{i<j} dH_{ij} \prod_{i=1}^{2j-1} dt_i / t_i. \tag{3.26}
\]

It follows from (3.23) that the \( 2j - 1 \times 2j - 1 \) (or \( 2j \times 2j \)) upper left blocks of \( H' \) have determinant \( t_i^2 \) (or 1) for \( j = 1, \ldots, \nu \) and that the \( j \times j \) upper left blocks have determinant 1 for \( j \geq 2\nu \). Independent variables are therefore the \( H_{ij} \) with \( i < j \) or \( i = j = 2k - 1 \) for \( k = 1, \ldots, \nu \). The other diagonal elements of \( H' \) are determined in terms of them by the previous condition.

We now come back to the original complex representation and express the differential element in (3.26) in terms of \( H \) alone. One can replace \( \prod_{i<j} dH_{ij} \) by the corresponding \( \prod dH_{ij} \) except for the first \( 2 \times 2 \) blocks along the diagonal.

If \((a, b)\) is the \( j \)th such block in \( H' \), the corresponding block in \( H \) is \((\xi a, \xi b)\), where \( u = \xi (a + c), \xi = \frac{1}{2} (a - c) + \sqrt{3} b \). From (3.23) it follows that \( a = \xi a' \) and \( c = \xi c' \), where \( a', c' \) depend only on \( Y \), as well as \( b \). In particular, for fixed \( Y \):
\[
2 \frac{dt_i}{t_i} = \frac{da}{a} - \frac{dc}{c} = \frac{d(a - c)}{a + c}. \tag{3.27}
\]

The contribution of this \( 2 \times 2 \) block to the differential element in (3.26) is then easily seen to be
\[
db \frac{dt_i / t_i}{t_i} = d\xi \xi dt_i / 4u. \tag{3.28}
\]

Therefore
\[
J = \frac{\Omega_0}{2^{N(2\pi)^{2^{N-2}\nu}}} \int \exp [-\text{Tr} \ S' S] d\mu (H), \tag{3.29}
\]
where
\[
d\mu (H) = \prod_{i<j} dH_{ij} \prod_{i=1}^{2j-1} dH_{2i, 2i-1}. \tag{3.30}
\]

We can now integrate over the last \( N - 2\nu \) rows and columns of \( H \) by the same induction procedure as in Sec. 1. We use (1.28) for \( n = N \) (\( x_n \) real). The integration over the last row and column is then straightforward and gives a factor
\[
\exp \left[ -\frac{1}{2} \sum_{i<j} (|x_i|^2)^{1/N} \prod_{i<j} (|x_i - x_j|)^{-1} \right]. \tag{3.31}
\]

After \( N - 2\nu \) similar steps, we obtain
\[
J = \frac{\Omega_0 \pi^{(N-2\nu)(N+2\nu-1)/4}}{2^N (2\pi)^{2^{N-2}\nu}(N-2\nu)! (4\pi)^{2\nu}} \prod_{i<j} |x_i - x_j| \times \exp \left[ -\sum_{i<j} |x_i|^2 \prod_{i<j} (|x_i - x_j|)^{-1} \right]. \tag{3.32}
\]

where \( F(x_i, \ldots, x_{2\nu}) \) is the value of the integral
\[
\text{exp} \left[ -\text{Tr} \ S' S \right] d\mu (H) \text{ when } S \text{ is a } 2\nu \times 2\nu \text{ matrix without real eigenvalues for } \nu > 0 \text{ and } F_0 = 1. \text{ Substituting (3.29) into (3.6), we obtain}
\]
\[
P_N (x_1, \ldots, x_N) = \frac{\Omega_0 \pi^{(N-2\nu)(N+2\nu-1)/4}}{2^N (N-2\nu)! (4\pi)^{2\nu}} \prod_{i<j} |x_i - x_j| \times \exp \left[ -\sum_{i<j} |x_i|^2 \prod_{i<j} (|x_i - x_j|)^{-1} \right]. \tag{3.33}
\]

In particular, for \( \nu = 0 \), all the eigenvalues are real: \( x_i = r_i \). The next \( N \) eigenvalues
\[
P_N (r_1, \ldots, r_N) = \frac{\Omega_0 \pi^{N(N-1)/4}}{2^N N!} \prod_{i<j} |r_i - r_j| \times \exp \left[ -\sum_{i<j} r_i^2 \right]. \tag{3.34}
\]

For \( \nu \neq 0 \), \( F_\nu \) cannot be calculated by the previous method. One can obtain a recursion formula analogous to (1.28), though somewhat less simple. However, the integration over the last two rows and columns of \( H \) does not lead to elementary functions as previously and the induction procedure breaks down completely.

Therefore we are able to determine \( P_N \) explicitly only in the particular case where all eigenvalues...
are real. In the general case, we have extracted the dependence of $P_N$ on the real eigenvalues, and the complex eigenvalues still appear in a factor $F$, for which we have only an intractable integral representation.

CONCLUSION

The results of our investigation are essentially contained in Eqs. (1.35), (1.40), and (1.42) for the complex case; (2.27) for the quaternion case; (3.30) for the real case. They are remarkably simple in the complex case, where, in particular, the eigenvalue density tends to a constant $\rho(x) \to \rho = 1/2\pi$ as $N \to \infty$. The quaternion case involves essentially technical problems. In the real case, however, one meets major difficulties which seem to come from the fact that the real field is not algebraically closed.

We have considered here only a restricted class of ensembles. One could generalize by keeping the same algebraic set of matrices and defining other measures $d\mu'$ instead of $d\mu$ defined in (0.1). For instance, if $d\mu'(S) = \varphi(S)d\mu(S)$, where $\varphi$ is a polynomial in $\text{Tr} (S')$ ($r = 1, \ldots, N$) (if $r' > N$, $\text{Tr} S'$ is a polynomial in $\text{Tr} S'$, $r \leq N$), then nothing is changed in the calculation of $P_N$, because $\varphi(S)$ depends only on $A$. However the induction procedure by which we performed the integration on $X$ (or $H$) seems to be a very specific property of $\text{exp} \{-\text{Tr} S'A\}$.

One could also consider other algebraic sets of matrices, for instance Dyson's $V$ ensembles (5) with measures analogous to (0.1). Nothing new appears in the complex case where $V$ is identical with $Z$. In the real case, $V$ is the set of complex symmetric matrices; the calculation of $P_N$ seems to be extremely complicated, and the simplest case of $2 \times 2$ matrices is not encouraging. In the quaternion case, we have not reached any conclusion.

ACKNOWLEDGMENTS

I wish to express my gratitude to Professor E. P. Wigner for his kind interest in this work and for enlightening conversations. I am also grateful to Princeton University for a Jane Eliza Procter Fellowship.