PROBLEM 1. (35 points.) Fano’s inequality gives a lower bound on the probability of error in terms of the conditional entropy. In this problem we will derive an upper bound on the error probability of a maximum-aposteriori-probability (MAP) decoder in terms of the conditional entropy.

Let $U$ and $V$ be random variables with joint distribution $p_{UV}$. We will think of $U$ as being the message, and $V$ the received symbol. Let $\hat{U}$ be the MAP estimate of $U$ from $V$.

Given that $V = v$ is received, the MAP decoder will decide on a $\hat{u}$ for which

$$\Pr(U = \hat{u} | V = v) \geq \Pr(U = u | V = v) \quad \text{for all } u.$$  

(a) (5 points) Show that when $V = v$, the MAP decoder makes an error with probability

$$\Pr(U \neq \hat{U} | V = v) = 1 - \max_u p_{U|V}(u|v).$$

(b) (5 points) Show that

$$\Pr(\hat{U} \neq U) = \sum_v p_V(v) [1 - \max_u p_{U|V}(u|v)].$$

(c) (15 points) Show that for any random variable $W$,

$$H(W) \geq \sum_w p_W(w) (1 - p_W(w)) \log e \geq [1 - \max_w p_W(w)] \log e.$$  

[Hint: $\ln z \leq z - 1$.]

(d) (5 points) Show that

$$H(U|V = v) \geq [1 - \max_u p_{U|V}(u|v)] \log e.$$  

(e) (5 points) Show that

$$\Pr(\hat{U} \neq U)(\log e) \leq H(U|V).$$
Problem 2. (35 points) Consider a binary, stationary, Markov source described by

\[ \Pr(X_{k+1} = 0 \mid X_k = 0) = \Pr(X_{k+1} = 1 \mid X_k = 1) = \alpha \]

where \(0 < \alpha < 1\).

(a) (5 points) Find the entropy rate of this source.

Given a sequence \(X_1, X_2, \ldots\), we can think of it as an alternating series of repetitions. For example if

\[ X_1, X_2, \ldots = 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, \ldots \]

it can be thought of as 0 repeated 3 times, 1 repeated 2 times, 0 repeated 1 time, 1 repeated 5 times, etc. Let \(R_1, R_2, \ldots\) be the lengths of these repetitions. In the example above, these are 3, 2, 1, 5, \ldots

(b) (5 points) Argue that \(R_1, R_2, \ldots\), form an i.i.d. sequence.

(c) (7 points) Find the probability distribution of \(R_k\).

(d) (10 points) Find the expectation \(E[R_1]\), and the entropy \(H(R_1)\). You may find the formula \(\sum_{k=0}^{\infty} k\alpha^k = \alpha/(1 - \alpha)^2\) useful.

(e) (8 points) The sequence \(X_1, X_2, \ldots\) can be described by first describing \(X_1\) using 1 bit and then describing \(R_1, R_2, \ldots\). Suppose the sequence \(R_1, R_2, \ldots\) is efficiently encoded into \(\mathcal{H}(R)\) bits per symbol.

How many bits per symbol does this method use in encoding the sequence \(X_1, X_2, \ldots\)? How does this compare to \(\mathcal{H}(X)\) found in (a)?
PROBLEM 3. (45 points) Given a stationary source $X_1, X_2, \ldots$, define

$$A_n(X) = \frac{1}{n} H(X_{n+1}, \ldots, X_{2n} | X_1, \ldots, X_n).$$

(a) (15 points) Show that $A_n(X) \geq A_{n+1}(X)$.

(b) (5 points) Find upper and lower bounds on $A_n(X)$ of the form

$$H(X_k | X_1, \ldots, X_{k-1}) \leq A_n(X) \leq H(X_m | X_1, \ldots, X_{m-1})$$

where $k$ and $m$ are appropriately chosen.

(c) (5 points) Show that

$$\lim_{n \to \infty} A_n(X) = H(X)$$

where $H(X)$ is the entropy rate of the source.

(d) (15 points) Given $n$, consider a source coding scheme that operates as follows:

1. The source output $X_1, X_2, \ldots$, is parsed into blocks of length $n$. The first block is $Y_1 = (X_1, \ldots, X_n)$, the second $Y_2 = (X_{n+1}, \ldots, X_{2n})$, the third, $Y_3 = (X_{2n+1}, \ldots, X_{3n})$, etc. Let $\mathcal{Y}$ denote the set of values of $Y_k$.

2. The first block $Y_1$ is coded in some uniquely decodable way, e.g., by using $\lceil \log |\mathcal{Y}| \rceil$ bits.

3. For $k \geq 1$, the $k+1$st block $Y_{k+1}$ is encoded into a codeword which may depend on $Y_k$. In other words, for each $v \in \mathcal{Y}$ we have a uniquely decodable code $C_v : \mathcal{Y} \to \{0, 1\}^*$, and to encode $Y_{k+1}$ we use the code $C_{Y_k}$.

Note that the long-term performance of this scheme will be determined by the performance of step 3.

Let $L_n$ be the average number of bits per source letter emitted by step 3. Find a lower bound to $L_n$ that applies to the scheme described above. [Hint: consider the quantity $A_n$.]

(e) (5 points) Find an upper bound on the minimum possible $L_n$. 