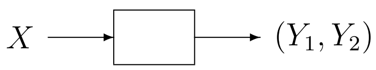
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 22	Information Theory and Coding
Homework 9	November 29, 2007

PROBLEM 1. Show that among all non-negative random variables with mean λ the exponential random variable has the largest differential entropy. Hint: let $p(x) = e^{-x/\lambda}/\lambda$ be the density of the exponential random variable and let $q(x)$ be some other density with mean λ. Consider $D(q\|p)$ and mimic the proof in the class for the maximal entropy of the Gaussian.

PROBLEM 2.



Consider the ordinary Shannon Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

with a power constraint P on X, and (Z_1, Z_2) a Gaussian zero mean random vector with covariance K, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

Find the capacity C for

(a) $\rho = 1$.

(b) $\rho = 0$.

(c) $\rho = -1$.

PROBLEM 3. Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

where

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right),$$

and there is a power constraint $E(X_1^2 + X_2^2) \leq 2P$. Assume that $\sigma_1^2 > \sigma_2^2$.

(a) Suppose we use the capacity achieving distribution as input. At what power does the channel stop behaving like a single channel with noise variance σ_2^2, and begin behaving like a pair of channels?

(b) Let $C_1(P)$ be the capacity of the pair of gaussian channels when the input is contrained to have a power not exceeding $2P$. Let $C_2(P) = I(X_1, X_2; Y_1, Y_2)$ when both X_1 and X_2 are independent gaussian random variables with variance equal to P. Show that $C_1(P) - C_2(P)$ tends to zero as P/σ_1^2 tends to infinity.

Problem 4. Consider a vector Gaussian channel described as follows:

$$\begin{aligned} Y_1 &= x + Z_1 \\ Y_2 &= Z_2 \end{aligned}$$

where x is the input to the channel constrained in power to P; Z_1 and Z_2 are jointly Gaussian random variables with $E[Z_1] = E[Z_2] = 0$, $E[Z_1^2] = E[Z_2^2] = \sigma^2$ and $E[Z_1 Z_2] = \rho\sigma^2$, with $\rho \in [-1, 1]$, and independent of the channel input.

(a) Consider a receiver that discards Y_2 and decodes the message based only on Y_1. What rates are achievable with such a receiver?

(b) Consider a receiver that forms $Y = Y_1 - \rho Y_2$, and decodes the message based only on Y. What rates are achievable with such a receiver?

(c) Find the capacity of the channel and compare it to the part (b).

Problem 5. Consider an additive noise channel with input $x \in \mathbb{R}$, and output

$$Y = x + Z$$

where Z is a real random variable independent of the input x, has zero mean and variance equal to σ^2.

In this problem we prove in two different ways that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance. Let $\mathcal{N}_{\sigma^2}$ denote the Gaussian density with zero mean and variance σ^2.

First Method: Let X be a Gaussian random variable with zero-mean and variance P. Let $\mathcal{N}_P$ denote its density $\mathcal{N}_P(x) = \frac{1}{\sqrt{2\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}}}$.

(a) Show that

$$I(X;Y) = H(X) - H(X - \alpha Y|Y)$$

for any $\alpha \in \mathbb{R}$.

(b) Observe that

$$H(X - \alpha Y) \leq \frac{1}{2} \log 2\pi e E((X - \alpha Y)^2)$$

for any $\alpha \in \mathbb{R}$.

(c) Deduce from (a) and (b) that

$$I(X;Y) \geq H(X) - \frac{1}{2} \log 2\pi e E((X - \alpha Y)^2)$$

for any $\alpha \in \mathbb{R}$.

(d) Show that

$$E((X - \alpha Y)^2) \geq \frac{\sigma^2 P}{\sigma^2 + P}$$

with equality if and only if $\alpha = \frac{P}{P+\sigma^2}$.

(e) Deduce from (c) and (d) that

$$I(X;Y) \geq \frac{1}{2} \log\left(1 + \frac{P}{\sigma^2}\right)$$

and conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.

Second Method:

(a) Denote the input probability density by p_X. Verify that

$$I(X;Y) = \iint p_X(x) p_Z(y-x) \ln \frac{p_Z(y-x)}{p_Y(y)}\, dx dy \quad \text{nats.}$$

where p_Y is the probability density of the output when the input has density p_X.

(b) Now set $p_X = \mathcal{N}_P$. Verify that

$$\tfrac{1}{2}\ln\big(1 + P/\sigma^2\big) = \iint p_X(x) p_Z(y-x) \ln \frac{\mathcal{N}_{\sigma^2}(y-x)}{\mathcal{N}_{P+\sigma^2}(y)}\, dx dy.$$

(c) Still with $p_X = \mathcal{N}_P$, show that

$$\tfrac{1}{2}\ln(1 + P/\sigma^2) - I(X;Y) \leq 0.$$

[Hint: use (a) and (b) and $\ln t \leq t - 1$.]

(d) Show that an additive noise channel with noise variance σ^2 and input power P has capacity at least $\frac{1}{2}\log_2(1 + P/\sigma^2)$ bits per channel use. Conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.

Problem 6. Consider an additive noise channel $Y = X + Z$ where X is the input of the channel, Y is the output of the channel and Z is the noise. The set of inputs to the channel are *non-negative* real numbers. Furthermore, the channel input is constrained in its average value: a codeword $\mathbf{x} = (x_1, \ldots, x_n)$ has to satisfy

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leq P.$$

The noise Z is independent of the input X, and has the exponential distribution with $E[Z] = 1$, i.e.,

$$f_Z(z) = \begin{cases} \exp(-z) & z \geq 0 \\ 0 & \text{else.} \end{cases}$$

(a) The capacity of this channel is given by

$$C = \max_{\substack{X: E[X] \leq P \\ X \text{ is non-negative}}} I(X;Y).$$

Express the mutual information in terms of the differential entropy of Y and the differential entropy of Z.

(b) What is the differential entropy of Z?

(c) For a random variable X that satisfies the input constraints, what are the constraints on the range and the expectation of Y? Find the maximum possible differential entropy of Y subject to these constraints. Hence show that the capacity is upper bounded by

$$C \leq \log(1 + P).$$

(d) Find the distribution on X that gives an exponential distribution for $Y = X + Z$

$$f_Y(y) = \mu e^{-\mu y} \quad \text{for } y \geq 0$$

[Use Laplace transforms to compute this distribution.]

(e) Conclude that the upper bound of part (c) is actually an equality, i.e.,

$$C = \log(1 + P).$$