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Handout 1 Information Theory and Coding
Homework 1 September 25, 2007

PROBLEM 1. Three events E_1, E_2 and E_3, defined on the same space, have probabilities $P(E_1) = P(E_2) = P(E_3) = 1/4$. Let E_0 be the event that one or more of the events E_1, E_2, E_3 occurs.

(a) Find $P(E_0)$ when:

(1) The events E_1, E_2 and E_3 are disjoint.

(2) The events E_1, E_2 and E_3 are independent.

(3) The events E_1, E_2 and E_3 are in fact three names for the same event.

(b) Find the maximum value $P(E_0)$ can assume when:

(1) Nothing is known about the independence or disjointness of E_1, E_2, E_3.

(2) It is known that E_1, E_2 and E_3 are pairwise independent, i.e., that the probability of realizing both E_i and E_j is $P(E_i)P(E_j)$, $1 \leq i \neq j \leq 3$, but nothing is known about the probability of realizing all three events together.

PROBLEM 2. A dishonest gambler has a loaded die which turns up the number 1 with probability 2/3 and the numbers 2 to 6 with probability 1/15 each. Unfortunately, he has left his loaded die in a box with two honest dice and can not tell them apart. He picks one die (at random) from the box, rolls it once, and the number 1 appears. Conditional on this result, what is the probability that he picked up the loaded die? He now rolls the dice once more and it comes up 1 again. What is the possibility after this second rolling that he has picked up the loaded die?

PROBLEM 3. This is a version of the Monty Hall problem — There are three closed doors. You are told that there is a car behind one of the doors and a goat behind each of the other two doors. Your objective is to get the car. You pick one of the doors at random with a uniform probability.

(a) What is the probability that you get the car?

After you make your choice, the doorkeeper opens one of the remaining two doors with a goat behind it. You are now allowed to make a choice between the door you already picked and the other door which is unopened.

(b) What is the probability that you get the car if you decide to choose the other unopened door?

PROBLEM 4. Let X and Y be two random variables.

(a) Prove that the expectation of the sum of X and Y, $E[X + Y]$, is equal to the sum of the expectations, $E[X] + E[Y]$.

(b) Prove that if X and Y are independent, then X and Y are also uncorrelated (by definition X and Y are uncorrelated if $E[XY] = E[X]E[Y]$). Find an example in which X and Y are statistically dependent yet uncorrelated.

(c) Prove that if X and Y are independent, then the variance of the sum $X+Y$ is equal to the sum of variances. Is this relationship valid if X and Y are uncorrelated but not independent?

PROBLEM 5. Suppose we flip a fair coin until it comes 'heads', let N be the number of flips we have made (I.e., $N=1$ if we get heads in our first try, $N=2$ if we first get a tail then a head, etc.).

(a) Find $\Pr\big(N=n \mid N\in\{n,n+1\}\big)$ for $n=1,2,\ldots$.

Suppose N is as above, and two boxes are prepared, one containing 3^{N-1} francs and the other containing 3^N francs. You have no idea which box contains which amount, neither do you know the value of N. You are allowed to open one of the boxes (lets call it the 'chosen box') and count the amount of money in it, and you have the option to either (i) take the money, or (ii) take the other box.

(b) Suppose you find 1 franc in the chosen box. How much money is there in the other box?

(c) Suppose you find 3^n francs in the chosen box, for some $n=1,2,\ldots$. Find the expected amount of money in the other box.

(d) If you solved parts (b) and (c) right, you should conclude that an expectation maximizer would take the other box no matter how much money he finds in the chosen box. In this case he can simply take the other box without even opening the chosen box. But then we are in the original situation with the 'other box' playing the role of the 'chosen box', so one should switch back, but then ....

Explain this 'other box is always better' paradox.

PROBLEM 6. A source has an alphabet of 4 letters. The probabilities of the letters and two possible sets of binary code words for the source are given below:

Letter	Prob.	Code I	Code II
a_1	0.4	1	1
a_2	0.3	01	10
a_3	0.2	001	100
a_4	0.1	000	1000

For *each* code, answer the following questions (no proofs or numerical answers are required).

(a) Is the code instantaneous?

(b) Is the code uniquely decodable?

PROBLEM 7. Design prefix-free codes with minimal expected length for the following source distributions. Explicitly specify the mapping from the source letter to the codeword and compute the expected length of the codeword. Compare this expected length with the entropy of the source.

(a) $\mathcal{A}=\{a,b,c,d,e\}$ with $\Pr(a)=\frac{1}{2}$, $\Pr(b)=\frac{1}{4}$, $\Pr(c)=\frac{1}{8}$, $\Pr(d)=\Pr(e)=\frac{1}{16}$.

(b) $\mathcal{A}=\{a,b,c,d,e\}$ with $\Pr(a)=\Pr(b)=0.1$, $\Pr(c)=\Pr(d)=0.2$, $\Pr(e)=0.4$.

(c) $\mathcal{A}=\{a,b,c,d,e\}$ with $\Pr(a)=\Pr(b)=\Pr(c)=\Pr(d)=\Pr(e)=0.2$.