MIDTERM SOLUTIONS

Wednesday, November 14, 2007, 10:15 - 13:15

Problem 1 [Hypothesis Testing – 20 pts]

(a) The MAP rule for the binary case

$$\frac{p_{Y|X}(y|1)}{p_{Y|X}(y|-1)} \underset{\hat{X}=-1}{\overset{\hat{X}=1}{\gtrless}} \frac{p_X(-1)}{p_X(1)} = 1. \tag{1}$$

We have,

$$\begin{aligned}\frac{p_{Y|X}(y|1)}{p_{Y|X}(y|-1)} &= \frac{\Pr(Z=W)p(y|1,Z=W)+\Pr(Z=W+1)p(y|1,Z=W+1)}{\Pr(Z=W)p(y|-1,Z=W)+\Pr(Z=W+1)p(y|-1,Z=W+1)}\\ &= \frac{1/2[p(y|1,Z=W)+p(y|1,Z=W+1)]}{1/2[p(y|-1,Z=W)+p(y|-1,Z=W+1)]}\\ &= \frac{f_W(y-1)+f_W(y-2)}{f_W(y+1)+f_W(y)}. \qquad (2)\end{aligned}$$

By visual inspection of Figure 1, it is seen that the right-hand side of (2) is > 1 when $y > 1$ (since in this case $f_W(y+1) = 0$ and $f_W(y-1) > f_W(y)$). Similarly, the right-hand side of (2) is < 1 when $y < 0$ (since in this case $f_W(y-2) = 0$ and $f_W(y) > f_W(y-1)$). Therefore it is sufficient to consider $0 \le y \le 1$ to find the threshold. In this case we have

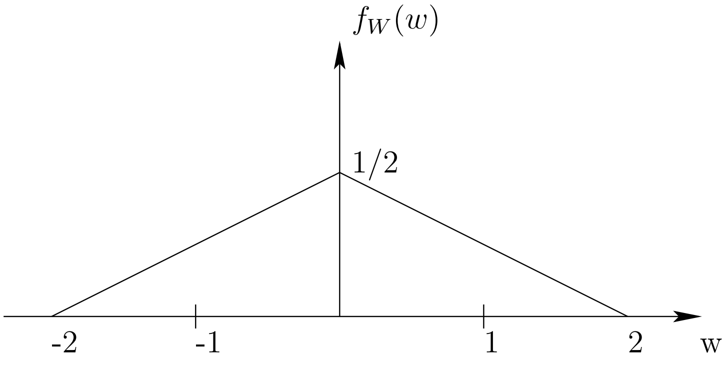

Figure 1:

$$1 = \frac{p_{Y|X}(y|1)}{p_{Y|X}(y|-1)} = \frac{1/2-|y-1|/4+1/2-|y-2|/4}{1/2-|y+1|/4+1/2-|y|/4} \tag{3}$$

$$= \frac{1/2-(1-y)/4+1/2-(2-y)/4}{1/2-(y+1)/4+1/2-y/4} \tag{4}$$

$$= \frac{1-(3-2y)/4}{1-(2y+1)/4}. \tag{5}$$

Solving (5) we get $y = 1/2$.

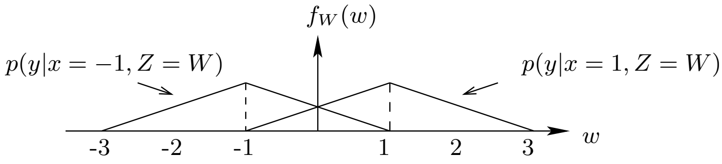

Figure 2:

(b) First consider the case $Z = W$. By symmetry (see Figure 2), it is easy to argue that in this case the error probability $P_{e,0}(t)$ is symmetric in t, and the threshold minimizing the error probability is $t = 0$. More precisely, $P_{e,0}(t)$ is given by

$$\begin{aligned} P_{e,0}(t) &= \frac{1}{2}\int_t^{\infty} p(t|x=-1,Z=W)dt + \frac{1}{2}\int_{-\infty}^{t} p(t|x=1,Z=W)dt \\ &= \frac{1}{2}\int_t^{\infty} f_W(t+1)dt + \frac{1}{2}\int_{-\infty}^{t} f_W(t-1)dt. \end{aligned}$$

Computing this integral for positive values of t we get

$$P_{e,0}(t) = \begin{cases} \frac{1}{16}[(1-t)^2 + (1+t)^2] & \text{if } 0 \le t < 1 \\ \frac{1}{2}[1 - \frac{(3-t)^2}{8}] & \text{if } 1 \le t < 3\,. \\ \frac{1}{2} & \text{if } 3 \le t \end{cases} \tag{6}$$

(You do not have to the compute (6) exactly, as long as you make the following

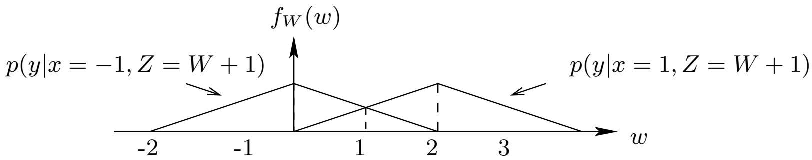

Figure 3:

observations:) By symmetry we have $P_{e,0}(t) = P_{e,0}(-t)$. Also note that $P_{e,0}(t)$ is increasing in $|t|$. Now consider the case $Z = W + 1$. We see in Figure 3 that the symmetry is preserved and the densities are merely shifted right by 1, therefore it is easily argued that $P_{e,1}(t)$ is of the same shape as $P_{e,0}(t)$, symmetric around $t = 1$ and increasing in $|t-1|$. Therefore the functions $P_{e,0}(t)$ and $P_{e,1}(t)$ take the form in Figure 4. It is seen in this figure that $\max\{P_{e,0}, P_{e,1}\}$ is minimized when $P_{e,0}(t) = P_{e,1}(t)$. Due to symmetry, equality is attained at $t = 1/2$.

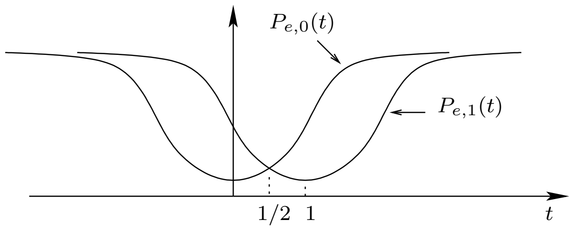

Figure 4:

Problem 2 [Proper Vectors - 20pts]

(a) We can write:

$$h_{>,\mathcal{F}}(f) = \frac{1}{2} + \frac{1}{2}\text{sign}(f)$$

where

$$\text{sign}(f) = \begin{cases} 1 & \text{for } f > 0 \\ 0 & \text{for } f = 0 \\ -1 & \text{for } f < 0 \end{cases}$$

Then, we obtain:

$$\begin{aligned} \hat{x}_{\mathcal{F}}(f) &= \sqrt{2}x_{\mathcal{F}}(f)h_{>,\mathcal{F}}(f) \\ &= \sqrt{2}x_{\mathcal{F}}(f)\left[\frac{1}{2} + \frac{1}{2}\text{sign}(f)\right] \\ &= \frac{x_{\mathcal{F}}(f)}{\sqrt{2}} + \frac{x_{\mathcal{F}}(f)}{\sqrt{2}}\text{sign}(f) \end{aligned}$$

The first term of the last line is symmetric in f (Fourier transform of a real-valued signal is symmetric). So the second term is anti-symmetric, so its inverse Fourier transform is a purelly imaginary signal. Hence, by taking the inverse Fourier transform, $\hat{x}(t)$ equals $\frac{x(t)}{\sqrt{2}}$ plus an imaginary term. So $x(t) = \sqrt{2}\text{Re}\left\{\hat{x}(t)\right\}$.

(b) $\hat{Z}(f) = \sqrt{2}Z_{\mathcal{F}}(f)h_{>,\mathcal{F}}(f)$ implies that

$$\hat{Z}(t) = \sqrt{2}\int_{-\infty}^{+\infty} h_{>}(\alpha)Z(t-\alpha)d\alpha$$

Then, the pseudocovariance of $\hat{Z}(t)$ is

$$\begin{aligned} \mathbb{E}\left[\hat{Z}(t)\hat{Z}(s)\right] &= \mathbb{E}\left[\sqrt{2}\int_{-\infty}^{+\infty} h_{>}(\alpha)Z(t-\alpha)d\alpha\sqrt{2}\int_{-\infty}^{+\infty} h_{>}(\beta)Z(s-\beta)d\beta\right] \\ &= 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_{>}(\alpha)h_{>}(\beta)\mathcal{R}_Z(t-\alpha-s+\beta)d\alpha d\beta \\ &= 2\int_{\alpha}\int_{\beta} h_{>}(\alpha)h_{>}(\beta)\int_{-\infty}^{+\infty} \mathcal{S}_Z(f)e^{j2\pi f(t-\alpha-s+\beta)}df \\ &= 2\int_{f} \mathcal{S}_Z(f)e^{j2\pi f(t-s)}h_{>,\mathcal{F}}(f)h_{>,\mathcal{F}}(-f)df \\ &= 0, \end{aligned}$$

since $h_{>,\mathcal{F}}(f)h_{>,\mathcal{F}}(-f) = 0$ for all frequencies except for $f = 0$. $\mathcal{R}_Z(\cdot)$ is the autocorrelation of $Z(t)$ and $\mathcal{S}_Z(f)$ is its Fourier transform. We have also used the fact that $h_{>,\mathcal{F}}(f) = \int_{\alpha} h_{>}(\alpha)e^{-j2\pi f\alpha}d\alpha$.

Hence the integral vanishes. Thus $\hat{Z}(t)$ is proper.

(c)

$$\begin{aligned}\mathbb{E}\left[Z_E(t)Z_E(s)\right] &= \mathbb{E}\left[\hat{Z}(t)e^{-j2\pi f_0 t}\hat{Z}(s)e^{-j2\pi f_0 s}\right] \\ &= e^{-j2\pi f_0(t+s)}\mathbb{E}\left[\hat{Z}(t)\hat{Z}(s)\right] \\ &= 0.\end{aligned}$$

(We could have simply argued that $Z_E(t)$ is proper since it is obtained from the proper process $\hat{Z}(t)$ via a linear tranformation).

(d) From point (c) we have

$$\begin{aligned}0 = \mathbb{E}\left[Z_E(t)Z_E(s)\right] &= \mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\} - \mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\}\right] \\ &\quad + j\mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\} + \mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\}\right]\end{aligned}$$

implies

$$\mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\}\right] = \mathbb{E}\left[\mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\}\right]$$

(e) We compute the autocorrelation of $Z_E(t)$:

$$\begin{aligned}\mathbb{E}\left[Z_E(t)Z_E^*(s)\right] &= \mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\} + \mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\}\right] \\ &\quad - j\mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\} - \mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\}\right]\end{aligned}$$

and observe that if the power spectral density of $Z_E(t)$ is symmetric (that is $S_Z(f_0 - f) = S_Z(f_0 + f)$), the autocorrelation of $Z_E(t)$ is real-valued. Thus

$$\mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\} - \mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\}\right] = 0$$

On the other hand, from point (d) we have

$$\mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\} + \mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\}\right] = 0$$

The last two expressions imply

$$\mathbb{E}\left[\mathrm{Re}\left\{Z_E(t)\right\}\mathrm{Im}\left\{Z_E(s)\right\}\right] = \mathbb{E}\left[\mathrm{Im}\left\{Z_E(t)\right\}\mathrm{Re}\left\{Z_E(s)\right\}\right] = 0$$

which says that the real and imaginary parts of $Z_E(t)$ are uncorrelated. But since they are Gaussian, this implies that they are independent.

Problem 3 [Viterbi Decoder – 20pts]

(a) The MAP rule is given by

$$\begin{aligned}
&\arg\max_{u_1^n} \mathrm{P}_{U_1^n|Y_1^{2n}}(u_1^n \mid y_1^{2n}) \\
&\quad = \arg\max_{u_1^n} \mathrm{P}_{Y_1^{2n}|U_1^n}(y_1^{2n} \mid u_1^n) \\
&\quad = \arg\max_{u_1^n} \Pi_{i=1}^n \mathrm{P}_{Y_{2i-1},Y_{2i}|U_i,U_{i-1}}(y_{2i-1}, y_{2i} \mid u_i, u_{i-1}) \\
&\quad = \arg\max_{u_1^n} \Pi_{i=1}^n \mathrm{P}_{Y_{2i-1}|U_i}(y_{2i-1} \mid u_i)\mathrm{P}_{Y_{2i}|U_i,U_{i-1}}(y_{2i} \mid u_i, u_{i-1}) \\
&\quad = \arg\max_{u_1^n} \Pi_{i=1}^n f(y_{2i-1}, u_i) f(y_{2i}, u_i + u_{i-1})
\end{aligned}$$

We have used the short notation x_1^n for the vector $(x_1, x_2, ...x_n)$.

(b) Taking the log of the above MAP rule, we get the MAP rule as

$$\arg\max_{u_1^n} \sum_{i=1}^{n} \log(f(y_{2i-1}, u_i)) + \log(f(y_{2i}, u_i + u_{i-1}))$$

The trellis section for the ith bit is given by

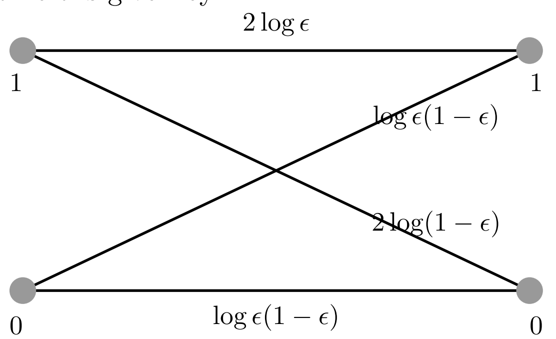

Figure 5: Branch metrics for $y_{2i-1} = 0, y_{2i} = 1$

(c) If the priors are not uniform then the trellis changes to

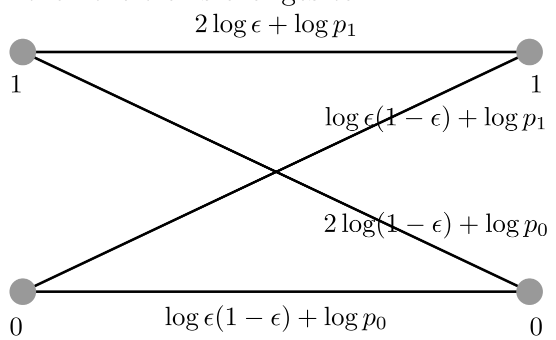

Figure 6: Branch metrics for $y_{2i-1} = 0, y_{2i} = 1$ with unequal priors

Problem 4 [Estimation – 20pts]

Let $\tilde{w}$ be any linear estimator and w_{opt} the linear estimator which satisfies the relation

$$\mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] = 0, \text{ for all } n$$

where $e_{opt}(k) = x(k) - \hat{x}_{opt}(k) = x(k) - w_{opt}(k) * y(k) = x(k) - \sum_n w_{opt}(n)y(k-n)$.
The estimator $\tilde{w}$ has an estimation error:

$$\tilde{e}(k) = x(k) - \tilde{w}(k) * y(k) = x(k) - \sum_n \tilde{w}(n)y(k-n).$$

Then

$$\begin{aligned}
\mathbb{E}\left[|\tilde{e}(k)|^2\right] &= \mathbb{E}\left[|\tilde{e}(k) - e_{opt}(k) + e_{opt}(k)|^2\right] \\
&= \mathbb{E}\left[|\tilde{e}(k) - e_{opt}(k)|^2\right] + \mathbb{E}\left[|e_{opt}(k)|^2\right] + 2\text{Re}\left\{\mathbb{E}\left[(\tilde{e}(k) - e_{opt}(k))^* e_{opt}(k)\right]\right\} \\
&= \mathbb{E}\left[|\tilde{e}(k) - e_{opt}(k)|^2\right] + \mathbb{E}\left[|e_{opt}(k)|^2\right] \\
&\quad +2\text{Re}\left\{\mathbb{E}\left[\left(x(k) - \sum_n \tilde{w}(n)y(k-n) - x(k) + \sum_n w_{opt}(n)y(k-n)\right)^* e_{opt}(k)\right]\right\} \\
&= \mathbb{E}\left[|\tilde{e}(k) - e_{opt}(k)|^2\right] + \mathbb{E}\left[|e_{opt}(k)|^2\right] \\
&\quad +2\text{Re}\left\{\sum_n (w_{opt}(n) - \tilde{w}(n))^* \mathbb{E}\left[e_{opt}(k)y^*(k-n)\right]\right\} \\
&= \mathbb{E}\left[|\tilde{e}(k) - e_{opt}(k)|^2\right] + \mathbb{E}\left[|e_{opt}(k)|^2\right] \\
&\geq \mathbb{E}\left[|e_{opt}(k)|^2\right].
\end{aligned}$$

where we have used the property of the optimal estimator, that is, $\mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] = 0$ for all n.

Therefore we get the "if" part directly due to the inequality $\mathbb{E}\left[|\tilde{e}(k)|^2\right] \geq \mathbb{E}\left[|e_{opt}(k)|^2\right]$. We get the "only if" part noticing that we need for optimality of *any* other estimator $\mathbb{E}\left[|\tilde{e}(k) - e_{opt}(k)|^2\right] = 0$, which means $\tilde{e}(k) = e_{opt}(k)$. This implies that $\sum_n \tilde{w}(n)y(k-n) = \sum_n w_{opt}(n)y(k-n)$, resulting in $\tilde{w}(n) = w_{opt}(n)$ for all n.

Problem 5 [Equalization – 20pts]

(a)

$$\begin{aligned} r_{xy}(n) &= \mathbb{E}\left[x(k)y^*(k-n)\right] \\ &= \mathbb{E}\left[x(k)\left(||p||\sum_s x^*(s)q^*(k-n-s)+z^*(k-n)\right)\right] \\ &= ||p||\sum_s \mathbb{E}\left[x(k)x^*(s)\right]q^*(k-n-s)+\mathbb{E}\left[x(k)z^*(k-n)\right] \\ &= ||p||\mathbb{E}\left[|x(k)|^2\right]q^*(-n) \\ &= ||p||\mathcal{E}_x q(n) \end{aligned}$$

since $\mathbb{E}\left[x(k)x^*(s)\right]=\mathcal{E}_x$ for $k=s$ and 0 otherwise.

$$\begin{aligned} r_{yy}(n) &= \mathbb{E}\left[y(k)y^*(k-n)\right] \\ &= \mathbb{E}\left[\left(||p||\sum_l x(l)q(k-l)+z(k)\right)\left(||p||\sum_s x^*(s)q^*(k-n-s)+z^*(k-n)\right)\right] \\ &= ||p||^2\sum_l\sum_s \mathbb{E}\left[x(l)x^*(s)\right]q(k-l)q^*(k-n-s)+\mathbb{E}\left[z(k)z^*(k-n)\right] \\ &= ||p||^2\sum_l \mathbb{E}\left[x(l)x^*(l)\right]q(k-l)q^*(k-n-l)+q(n)N_0 \\ &= ||p||^2\sum_l \mathcal{E}_x q(k-l)q(n-(k-l))+q(n)N_0 \\ &= ||p||^2\sum_l \mathcal{E}_x q(l)q(n-l)+q(n)N_0 \\ &= ||p||^2\mathcal{E}_x q(n)*q(n)+q(n)N_0. \end{aligned}$$

(b)

$$\begin{aligned} S_{xy}(D) &= \mathcal{D}\{r_{xy}(n)\} \\ &= \mathcal{D}\left\{||p||\mathcal{E}_x q(n)\right\} \\ &= ||p||\mathcal{E}_x Q(D) \end{aligned}$$

$$\begin{aligned} S_{yy}(D) &= \mathcal{D}\{r_{yy}(n)\} \\ &= \mathcal{D}\left\{||p||^2\mathcal{E}_x q(n)*q(n)+q(n)N_0\right\} \\ &= ||p||^2\mathcal{E}_x Q(D)Q(D)+Q(D)N_0 \\ &= ||p||^2\mathcal{E}_x Q^2(D)+Q(D)N_0 \end{aligned}$$

(c)

$$\begin{aligned}
0 = \mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] &= \mathbb{E}\left[(x(k) - w_{opt}(k) * y(k))y^*(k-n)\right] \\
&= \mathbb{E}\left[(x(k) - \sum_l w_{opt}(l)y(k-l))y^*(k-n)\right] \\
&= \mathbb{E}\left[x(k)y^*(k-n)\right] - \sum_l w_opt(l)\mathbb{E}\left[y(k-l)y^*(k-n)\right] \\
&= r_{xy}(n) - \sum_l w_{opt}(l)r_{yy}(n-l) \\
&= r_{xy}(n) - w_{opt}(n) * r_{yy}(n)
\end{aligned}$$

So $r_{xy}(n) = w_{opt}(n) * r_{yy}(n)$. Or in D-domain $S_{xy}(D) = W_{opt}(D)S_{yy}(D)$. The optimal filter $W_{opt}(D)$ is

$$\begin{aligned}
W_{opt}(D) &= \frac{S_{xy}(D)}{S_{yy}(D)} \\
&= \frac{||p||\mathcal{E}_x Q(D)}{||p||^2\mathcal{E}_x Q^2(D) + Q(D)N_0}
\end{aligned}$$