Advanced Digital Communications — EPFL, Winter Semester 2007
Nicolae Chiurtu — Wed, November 14, 2007

MIDTERM

Wednesday, November 14, 2007, 10:15 - 13:15
This exam has 5 problems and 100 points in total

- You have 3 hours.
- You are allowed to use 3 sheets of paper (6 pages) for reference.
- No other materials are allowed. No mobile phones or calculators are allowed.
- Do not spend too much time on each problem but try to collect as many points as possible.
- Write only what is relevant to the question!

Good Luck!

Name: ______________________________

Prob I	/ 20
Prob II	/ 20
Prob III	/ 20
Prob IV	/ 20
Prob V	/ 20
Total	/ 100

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Problem 1 [Hypothesis Testing – 20 pts]
Consider a channel whose input X and output Y are related by

$$Y = X + Z.$$

The input X is uniformly distributed on the set $\{-1, +1\}$, i.e.,

$$\Pr[X = -1] = \Pr[X = +1] = 1/2.$$

The noise Z is independent from X. Given the output Y, we want to guess X.

(*8 pts*) (a) Suppose that Z is given by

$$Z = \begin{cases} W & \text{with probability} \quad 1/2 \\ W + 1 & \text{with probability} \quad 1/2 \end{cases} \tag{1}$$

where W is a random variable with density

$$f_W(w) = \begin{cases} \frac{1}{2} - \frac{|w|}{4} & \text{if} \quad |w| \leq 2 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Find the decision rule that minimizes the probability of error.

(*12 pts*) (b) Suppose now that

$$Z = W \qquad \text{or} \qquad Z = W + 1, \tag{3}$$

where W is the same as in part (a), but (1) is not valid anymore. Instead, we want to find a decision rule that works well in both the cases in (3). For a given decision scheme H, define the error probability in each case by

$$P_{e,0}(H) = \Pr[\text{error}|Z = W]$$
$$P_{e,1}(H) = \Pr[\text{error}|Z = W + 1].$$

Find the decision rule that minimizes the error probability in the worst case. That is, find H that minimizes

$$\max\{P_{e,0}(H), P_{e,1}(H)\}. \tag{4}$$

Hint 1: Your decision rule should take the form of a threshold scheme.

Hint 2: What do the plots of $P_{e,0}(t)$ and $P_{e,1}(t)$ look like as a function of the threshold t of the decision scheme?

Problem 2 [Proper Vectors – 20pts]

(*4 pts*) (a) Consider an arbitrary real-valued signal $x(t)$ and define $\hat{x}(t)$ to be the signal with the Fourier transform

$$\hat{x}_{\mathcal{F}}(f) = \sqrt{2}x_{\mathcal{F}}(f)h_{>,\mathcal{F}}(f)$$

where

$$h_{>,\mathcal{F}}(f) = \begin{cases} 1 & \text{for } f > 0 \\ 1/2 & \text{for } f = 0 \\ 0 & \text{for } f < 0 \end{cases}$$

and $x_{\mathcal{F}}(f)$ is the Fourier transform of $x(t)$, $x_{\mathcal{F}}(f) = \mathcal{F}\{x(t)\}$.
Derive the relation that allows to go back from $\hat{x}(t)$ to $x(t)$, i.e., find $x(t)$ as a function of $\hat{x}(t)$.

(*4 pts*) (b) Let $\hat{Z}(f) = \sqrt{2}Z_{\mathcal{F}}(f)h_{>,\mathcal{F}}(f)$ where $Z_{\mathcal{F}}(f) = \mathcal{F}\{Z(t)\}$ and $Z(t)$ is a zero-mean, real-valued Gaussian and wide sense stationary process. Using the definition of a proper vector, show that $\hat{Z}(t) = \mathcal{F}^{-1}\{\hat{Z}(f)\}$ is proper.

(*4 pts*) (c) Show that $Z_E(t) = \hat{Z}(t)e^{-j2\pi f_0 t}$ is also proper. (f_0 is a positive real number.)

(*4 pts*) (d) Use the previous results to show that the real and imaginary components of $Z_E(t)$ have the same autocorrelation function.

(*4 pts*) (e) Assume that $S_Z(f_0 - f) = S_Z(f_0 + f)$ where $S_Z(f)$ is the Fourier transform of the autocorrelation function of $Z_E(t)$. Show that the real and imaginary parts of $Z_E(t)$ are independent.

Problem 3 [Viterbi Decoder – 20pts]
Let $\{U_1, \ldots, U_n\}$ be a sequence of i.i.d. random variables over $\{0, 1\}$ with probability distribution $\mathrm{P}(0) = \mathrm{P}(1) = 1/2$. Consider the following channel. The input and output of the channel are binary $\{0, 1\}$ with the following transition probabilities

$$\begin{aligned}
\mathrm{P}_{Y|X}(1 \mid 1) &= 1 - \epsilon, \\
\mathrm{P}_{Y|X}(0 \mid 1) &= \epsilon, \\
\mathrm{P}_{Y|X}(0 \mid 0) &= 1 - \epsilon, \\
\mathrm{P}_{Y|X}(1 \mid 0) &= \epsilon.
\end{aligned}$$

Let $f(x, y)$ denote the transition probability $\mathrm{P}_{Y|X}(y \mid x)$.

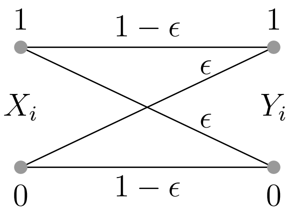

Figure 1: The graph showing transmission between source and receiver

Let $U_0 = 0$. For each bit U_i, $1 \leq i \leq n$ we transmit two bits $X_{2i-1} = U_i$ and $X_{2i} = U_i + U_{i-1}$, where the addition here means XOR (modulo 2 addition). Let the source vector be $\{U_1, \ldots, U_n\}$, the transmitted vector be $\{X_1, \ldots, X_{2n}\}$ and the received vector be $\{Y_1, \ldots, Y_{2n}\}$.

(*5 pts*) (a) Write the MAP rule for decoding the sequence $\{U_1 \ldots, U_n\}$ from $\{Y_1, \ldots, Y_{2n}\}$ in terms of the function f.

(*10 pts*) (b) Construct a Viterbi decoder for this MAP rule. Use log domain calculation. Let $y_{2i-1} = 0, y_{2i} = 1$. Draw the trellis section related to the bit U_i and compute the branch metrics in terms of ϵ.

(*5 pts*) (c) Consider the case where the priors are not uniform. Let $\mathrm{P}(0) = p_0$, $\mathrm{P}(1) = p_1 = 1 - p_0$. What are the new branch metrics?

Problem 4 [Estimation – 20pts]
Suppose we observe the sequence $\{y(k)\}$ and we want to estimate $\{x(k)\}$. For this purpose, we use a linear filter $w(k)$. Let $\hat{x}(k)$ be the estimate of $x(k)$:

$$\hat{x}(k) = w(k) * y(k).$$

Let $w_{opt}(k)$ be the optimal estimator according to the MMSE criterion, i.e. the estimator minimizing $\mathbb{E}\left[|e(k)|^2\right]$, where $e(k) = x(k) - \hat{x}(k)$.
Show that the optimal estimator $w_{opt}(k)$ is minimizing $\mathbb{E}\left[|e(k)|^2\right]$ *if and only if* the following condition is met:

$$\mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] = 0, \text{ for all } n$$

where $e_{opt}(k) = x(k) - \hat{x}_{opt}(k)$ and $\hat{x}_{opt}(k) = w_{opt}(k) * y(k)$.
Provide a complete proof, i.e., prove both "if" and "only if" parts.
Hint: start by evaluating the performance on an estimator which does not satisfy the condition above.

Problem 5 [Equalization – 20pts]
Consider the channel:

$$y(k) = ||p||x(k) * q(k) + z(k),$$

where the input symbols $\{x(k)\}$ are i.i.d. and independent of the noise samples $\{z(k)\}$. We denote the energy of the transmitted symbols by $\mathcal{E}_x = \mathbb{E}\left[|x(k)|^2\right]$. The noise samples $\{z(k)\}$ have the autocorrelation function $R_z(k) = q(k)N_0$.
The channel coefficients have the property that $(q(n))^* = q(-n)$, and p is a constant scalar.

(*12 pts*) (a) Using time domain calculations, compute $r_{xy}(n) = \mathbb{E}\left[x(k)y^*(k-n)\right]$ and $r_{yy}(n) = \mathbb{E}\left[y(k)y^*(k-n)\right]$ in terms of $q(n)$.

(*3 pts*) (b) Compute the cross spectrum and power spectrum $S_{xy}(D) = \mathcal{D}\{r_{xy}(n)\}$ and $S_{yy}(D) = \mathcal{D}\{r_{yy}(n)\}$ respectively.

(*5 pts*) (c) We convolve $y(k)$ with a filter $w(k)$ in order to estimate $x(k)$ and define the estimation error $e(k) = x(k) - w(k) * y(k)$. In order to minimize this error according to the MMSE criterion, we need to satisfy $\mathbb{E}\left[e_{opt}(k)y^*(k-n)\right] = 0$ for all n, where $e_{opt}(k) = x(k) - w_{opt}(k) * y(k)$. Use this condition to derive the optimal filter $w_{opt}(k)$. Using the previous parts of the exercise compute the optimal filter in D-transform domain.