Solutions to Homework 5

Problem 1

(a)

$$\begin{aligned}
\mathbb{E}[h_l(t_0)] &= \mathbb{E}[C_l e^{j(2\pi\frac{v}{\lambda}\cos\theta_l t+\phi_l)}] \\
&= \mathbb{E}[C_l]\mathbb{E}[e^{j(2\pi\frac{v}{\lambda}\cos\theta_l t)}]\mathbb{E}[e^{j\phi_l}] \\
\mathbb{E}[e^{j\phi_l}] &= \int_0^{2\pi} e^{j\phi_l}\frac{1}{2\pi}d\phi_l \\
&= [-\frac{j}{2\pi}e^{j\phi_l}]_0^{2\pi} = 0 \\
\mathbb{E}[h_l(t_0)] &= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{Var}[h_l(t_0)] &= \mathbb{E}[|h_l(t_0)|^2] - \underbrace{\mathbb{E}[h_l(t_0)]^2}_{=0 \text{ from (a)}} \\
&= \mathbb{E}[|C_l e^{j(2\pi\frac{v}{\lambda}\cos\theta_l t+\phi_l)}|^2] \\
&= \mathbb{E}[|C_l|^2]\underbrace{\mathbb{E}[|e^{j(2\pi\frac{v}{\lambda}\cos\theta_l t)}|^2]}_{=1}\underbrace{\mathbb{E}[|e^{j\phi_l}|^2]}_{=1} \\
&= \frac{\sigma^2}{L} \quad \text{(second moment of } C_l)
\end{aligned}$$

(c)

$$X_l = \frac{\sqrt{L}}{\sigma}h_l(t_0), \qquad \mathbb{E}[X_l] = \frac{\sqrt{N}}{\sigma}\underbrace{\mathbb{E}[h_l(t_0)]}_{=0} = 0, \qquad \mathrm{Var}(X_l) = \frac{L}{\sigma^2}\underbrace{\mathrm{Var}(h_l(t_0))}_{=\frac{\sigma^2}{L}} = 1$$

Remember that $h(t) = \sum_{l=1}^{L} h_l(t)$. At time $t = t_0$:

$$h(t_0) = \sum_{l=1}^{L} h_l(t_0) = \sum_{l=1}^{L} X_l\frac{\sigma}{\sqrt{L}} = \sigma\frac{1}{\sqrt{L}}\sum_{l=1}^{L} X_l$$

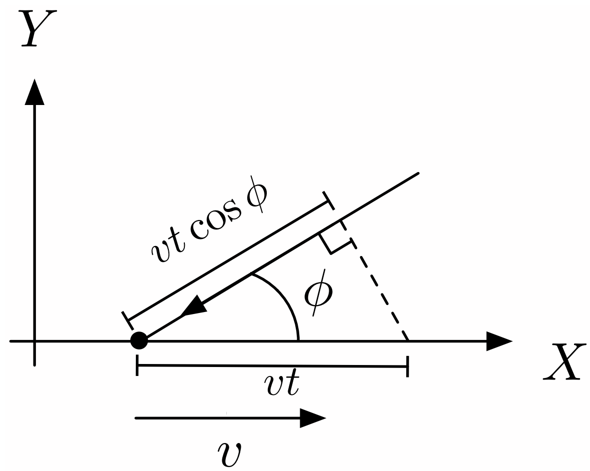
All X_i's are i.i.d since $h_i(t_0)$'s are composed of i.i.d variables. Using the central limit theorem we get:

$$\begin{aligned}
\lim_{L\to\infty}\frac{1}{\sqrt{L}}\sum_{l=1}^{L} X_l &\sim \mathcal{N}(0,1) \\
\lim_{L\to\infty}\sigma\frac{1}{\sqrt{L}}\sum_{l=1}^{L} X_l &\sim \mathcal{N}(0,\sigma^2)
\end{aligned}$$

Thus for large L, $h(t)$ behaves as a zero mean Gaussian random variable of variance σ^2.

Problem 7.3

The mobile receiver has a velocity if $v\cos\phi$ towards the transmitter. The received



electric field is hence

$$\begin{aligned} E[f,t,(r,\theta,\psi)] &= \frac{1}{r_0+vt\cos\phi}\Re[\alpha(\theta,\psi,f)e^{j2\pi f(t-\frac{r_0}{c}+\frac{vt\cos\phi}{c})}] \\ &= \frac{1}{r_0+vt\cos\phi}\Re[\alpha(\theta,\psi,f)e^{j2\pi f(1+\frac{v\cos\phi}{c})t}e^{j2\pi f\frac{r_0}{c}}] \end{aligned}$$

The Doppler shift is $f \to f(1+\frac{v\cos\phi}{c})$.

Problem 8.3

1. $\frac{H^*}{|H|}Y$ is an invertible mapping and is given by:

$$\begin{aligned} \frac{H^*}{|H|}Y &= \frac{H^*}{|H|}(HX+W) \\ &= \frac{|H|^2X}{|H|}+\frac{WH^*}{|H|} \\ &= |H|X+\frac{WH^*}{|H|} \end{aligned}$$

Since $|H|X$ is real, the real part of $\frac{H^*}{|H|}Y$ is a sufficient statistics for detection of X.

Now let's compute the distribution of $\frac{WH^*}{|H|}$:

$$\mathbb{E}\left[\frac{WH^*}{|H|}\frac{HW^*}{|H|}\right]=\mathbb{E}[|W|^2]=N_0.$$

We are interrested in the real part of the noise i.e. $Re\left(\frac{WH^*}{|H|}\right)$. So, it has variance of$\frac{N_0}{2}$. So $W\sim\mathbb{C}\mathcal{N}(0,\frac{N_0}{2})$.

2. The exact probability of error for a BPSK knowing H is given by:

$$P_{e/H} = Q\left(\frac{a|h|}{\sqrt{\frac{N_0}{2}}}\right)$$

The probability of error is given by the expectation over H:

$$P_e = \mathbb{E}_{|h|}[P_{e/H}] = \mathbb{E}_{|h|}\left[Q\left(\frac{a|h|}{\sqrt{\frac{N_0}{2}}}\right)\right]$$

Let's rewrite this expression in term of the $SNR = \frac{a^2}{N_0}$, we have $\frac{a}{\sqrt{\frac{N_0}{2}}} = \sqrt{2SNR}$. By replacing, we get :

$$P_e = \mathbb{E}_{|h|}\left[Q\left(|h|\sqrt{2SNR}\right)\right]$$

Let's evaluate this expression using $f_{|h|}(r) = 2re^{-r^2}$:

$$\begin{aligned}
\mathbb{E}_{|h|}\left[Q\left(|h|\sqrt{2SNR}\right)\right] &= \int_0^\infty 2re^{-r^2}\frac{1}{\sqrt{2\pi}}\int_{r\sqrt{2SNR}}^\infty e^{-\frac{y^2}{2}}dydr \\
&= \int_0^\infty \int_{r\sqrt{2SNR}}^\infty \frac{1}{\sqrt{2\pi}}2re^{-r^2}e^{-\frac{y^2}{2}}dydr \\
&= \int_0^\infty \int_0^{\frac{y}{\sqrt{2SNR}}} \frac{1}{\sqrt{2\pi}}2re^{-r^2}e^{-\frac{y^2}{2}}drdy \quad \text{(Change in the integration ord} \\
&= \int_0^\infty \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\int_0^{\frac{y}{\sqrt{2SNR}}} 2re^{-r^2}drdy \\
&= \int_0^\infty \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\left(1 - e^{-\frac{y^2}{\sqrt{2SNR}}}\right)dy \\
&= \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{y^2}{2}\frac{(1+SNR)}{SNR}}dy \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{SNR}{(1+SNR)}} = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{(1+SNR)}}\right)
\end{aligned}$$

3.

$$P_e = Q\left(\frac{a}{\sqrt{\frac{N_0}{2}}}\right) = Q(\sqrt{2SNR}).$$

4. For the flat fading channel :$P_e = \{0.146, 0.023, 0.0025, 0.000025, 2.5 \times 10^{-9}\}$.
   For the AWGN channel : $P_e = \{0.367, 0.000045, 3.7 \times 10^{-44}, 0, 0\}$.

Problem 8.4

1.

$$\begin{aligned} P_e &= \mathbb{E}_{||\mathbf{h}||}\left[Q\left(\frac{||\mathbf{h}||d_{min}}{2\sigma}\right)\right] \\ &\leq \mathbb{E}_{||\mathbf{h}||^2}\left[e^{-\frac{||\mathbf{h}||^2 d_{min}^2}{8\sigma^2}}\right] = \mathbb{E}_{||\mathbf{h}||^2}\left[e^{-\frac{\mathbf{h}^* d_{min}^2 \mathbf{h}}{8\sigma^2}}\right] \\ &= \frac{1}{|\mathbf{I} + \frac{d_{min}^2}{8\sigma^2}\mathbf{K_h}|} \quad \text{(Lemma from the course)} \\ &= \frac{1}{|\mathbf{I} + \frac{d_{min}^2}{8\sigma^2}\mathbf{\Lambda}|} \end{aligned}$$

where $\mathbf{\Lambda}$ is the eigenvalue matrix of $\mathbf{K_h}$.

2. $\mathbf{K_h}$ is positive definite, then $\mathbf{\Lambda}$ will have only strictly positive values. Hence

$$\begin{aligned} \frac{1}{|\mathbf{I} + \frac{d_{min}^2}{8\sigma^2}\mathbf{\Lambda}|} &= \frac{1}{\prod_{i=0}^{D-1}(1+\lambda_i SNR)} \\ &\leq \frac{1}{\prod_{i=0}^{D-1}(\lambda_i SNR)} \\ &\doteq \frac{1}{SNR^D} \end{aligned}$$

Hence the diversity order is D. We can see that the diversity order doesn't change when the $h(l)$ are not independent.

Problem 8.5

1. For one receiver, the error probability is:

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right) \approx \frac{1}{2}\left(1 - \left(1 - \frac{1}{2SNR}\right)\right) = \frac{1}{4SNR}$$

Using upper bound for two receive antennas:

$$\begin{aligned} P_e &= \mathbb{E}_h Q(\frac{||h||\sqrt{\mathcal{E}_x}}{2\sigma}) \\ &\leq \mathbb{E}_h e^{-||h||^2\mathcal{E}_x/8\sigma^2} \\ &= \frac{1}{(1+\frac{\mathcal{E}_x}{8\sigma^2})^2} \\ &\approx \frac{c}{SNR^2} \end{aligned}$$

If we define "$\doteq$" notation as

$$\lim_{\text{SNR}\to\infty} \frac{\log P_e(\text{SNR})}{\log \text{SNR}} = -d$$

written is shorthand as $P_e(\text{SNR}) \doteq \text{SNR}^{-d}$. Then $P_e \doteq \frac{1}{\text{SNR}^2}$ above.

2. When the flag is obstructing we have only one path from the transmitter to the receiver. Therefore,

$$P_e(.|\mathcal{F}=0) \doteq \frac{1}{SNR}$$

   In the absence of the flag, we have two indpendent paths from the transmitter to the receiver. Therefore,

$$P_e(.|\mathcal{F}=1) \doteq \frac{1}{SNR2}$$

3.

$$\begin{aligned} P_e &= \Pr(\mathcal{F}=1)P_e(.|\mathcal{F}=1) + \Pr(\mathcal{F}=0)P_e(.|\mathcal{F}=0) \\ &\doteq (1-q)\frac{1}{SNR2} + q\frac{1}{SNR} \\ &\doteq \frac{1}{SNR} \end{aligned}$$

Therefore the diversity order is 1.

Problem 8 .6

1. Let

$$\mathbf{x_A} = \mathbf{R}\begin{bmatrix} a \\ a \end{bmatrix}, \mathbf{x_B} = \mathbf{R}\begin{bmatrix} -a \\ a \end{bmatrix}$$
$$\mathbf{x_C} = \mathbf{R}\begin{bmatrix} -a \\ -a \end{bmatrix}, \mathbf{x_D} = \mathbf{R}\begin{bmatrix} a \\ -a \end{bmatrix}$$

   Now, by union bound :

$$P_e \le \mathbb{P}(\mathbf{x_A} \to \mathbf{x_B}) + \mathbb{P}(\mathbf{x_A} \to \mathbf{x_C}) + \mathbb{P}(\mathbf{x_A} \to \mathbf{x_D})$$

   Lets evaluate $\mathbb{P}(\mathbf{x_A} \to \mathbf{x_B})$. We can rewrite the channel using the sufficient statistics and $\mathbf{x_A}$ as the transmitted symbol:

$$\frac{h_1^*}{|h_1|}y_1 = x_{A1}|h_1| + z_1$$
$$\frac{h_2^*}{|h_2|}y_1 = x_{A2}|h_2| + z_2$$

where z_1 and z_2 are $\mathcal{N}(0, \frac{N_0}{2})$.

Let $\mathbf{V_A} = \begin{bmatrix} |h_1| x_{A1} \\ |h_2| x_{A2} \end{bmatrix}$ and $\mathbf{V_B} = \begin{bmatrix} |h_1| x_{B1} \\ |h_2| x_{B2} \end{bmatrix}$.

Now,

$$\mathbb{P}(\mathbf{x_A} \to \mathbf{x_B} | h_1, h_2) = Q\left(\frac{||\mathbf{V_A} - \mathbf{V_B}||}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{SNR(|h_1|^2 d_1^2 + |h_2|^2 d_2^2)}{2}}\right)$$

where $SNR = \frac{a^2}{N_0}$ and $\mathbf{d} = \frac{1}{a}(\mathbf{x_A} - \mathbf{x_B}) = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \end{bmatrix}$.

And

$$\begin{aligned} \mathbb{E}_{h_1,h_2}\left[\mathbb{P}(\mathbf{x_A} \to \mathbf{x_B} | h_1, h_2)\right] &\leq \mathbb{E}_{h_1,h_2}\left[e^{-\frac{SNR(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}{4}}\right] \\ &= \frac{1}{\left(1 + SNR\frac{|d_1|^2}{4}\right)} \frac{1}{\left(1 + SNR\frac{|d_2|^2}{4}\right)} \\ &\leq \frac{16}{|d_1 d_2|^2} \frac{1}{SNR^2} \end{aligned}$$

Let $\delta_{AB} = |d_1 d_2|^2$. We can repeat the calculus for $\mathbb{P}(\mathbf{x_A} \to \mathbf{x_C})$ and $\mathbb{P}(\mathbf{x_A} \to \mathbf{x_D})$. By analogy, we can define $\delta_{AC}$and δ_{AD}. Hence

$$P_e \leq 16\left(\frac{1}{\delta_{AB}} + \frac{1}{\delta_{AC}} + \frac{1}{\delta_{AD}}\right)\frac{1}{SNR^2}.$$

2. To get diversity order of 2, none of the δ_{AB}, δ_{AC} and δ_{AD} should be zero. We have

$$\delta_{AB} = \delta_{AD} = 4\sin^2 2\theta$$

and

$$\delta_{AC} = 16\cos^2 2\theta$$

It gives the following conditions:

$$\begin{aligned} \sin 2\theta = 0 \Rightarrow 2\theta &= \pm n\pi \\ \theta &= \pm\frac{n\pi}{2} = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi \\ \cos 2\theta = 0 \Rightarrow 2\theta &= \frac{\pi}{2}, \frac{3\pi}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4} \end{aligned}$$

So

$$\theta \neq \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}\}.$$