Homework 4

Problem 3

Consider the scalar discrete-time inter symbol interference channel considered in the class,

$$y_k = \sum_{n=0}^{\nu} p_n x_{k-n} + z_k, \quad k = 0, \dots, N-1, \tag{1}$$

where $z_k \sim \mathbf{C}\mathcal{N}(0, \sigma_z^2)$ and is i.i.d., independent of $\{x_k\}$. Let us employ a cyclic prefix as done in OFDM, *i.e.,*

$$x_{-l} = x_{N-1-l}, \quad l = 0, \dots, \nu.$$

As done in class given the cyclic prefix,

$$\mathbf{y} = \begin{bmatrix} y_{N-1} \\ \vdots \\ y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} p_0 & \dots & \dots & p_\nu & 0 & \dots & 0 & 0 \\ 0 & p_0 & \dots & p_{\nu-1} & p_\nu & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & p_0 & \dots & \dots & p_\nu \\ p_\nu & 0 & \dots & 0 & 0 & p_0 & \dots & p_{\nu-1} \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ p_1 & \dots & p_\nu & 0 & \dots & 0 & 0 & p_0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}}. \tag{2}$$

In the derivation of OFDM we used the property that

$$\mathbf{P} = \mathbf{F}^* \mathbf{D} \mathbf{F}, \tag{3}$$

where

$$\mathbf{F}_{p,q} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N}(p-1)(q-1)\right)$$

and $\mathbf{D}$ is the diagonal matrix with

$$\mathbf{D}_{l,l} = d_l = \sum_{n=0}^{\nu} p_n e^{-j\frac{2\pi}{N}nl}.$$

Using this we obtained

$$\mathbf{Y} = \mathbf{F}\mathbf{y} = \mathbf{D}\mathbf{X} + \mathbf{Z},$$

where $\mathbf{X} = \mathbf{F}\mathbf{x}$, $\mathbf{Z} = \mathbf{F}\mathbf{z}$. This yields the parallel channel result

$$\mathbf{Y}_l = d_l \mathbf{X}_l + \mathbf{Z}_l. \tag{4}$$

If the carrier synchronization is not accurate, then (1) gets modified as

$$y(k) = \sum_{n=0}^{\nu} e^{j2\pi f_0 k} p_n x_{k-n} + z_k, \quad k = 0, \dots, N-1 \tag{5}$$

where f_0 is the carrier frequency offset. If we still use the cyclic prefix for transmission, then (2) gets modified as

$$\underbrace{\begin{bmatrix} y(N-1) \\ \cdot \\ \cdot \\ y(0) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} p_0 e^{j2\pi f_0 (N-1)} & \dots & p_\nu e^{j2\pi f_0 (N-1)} & 0 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & \ddots & e^{j2\pi f_0 \nu} p_0 & \dots & e^{j2\pi f_0 \nu} p_\nu \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ e^{j2\pi f_0 0} p_1 & \dots & e^{j2\pi f_0 0} p_\nu & 0 & \dots & 0 & e^{j2\pi f_0 0} p_0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}} . \tag{6}$$

i.e.,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

Note that

$$\mathbf{H} = \mathbf{S}\mathbf{P},$$

where $\mathbf{S}$ is a diagonal matrix with $\mathbf{S}_{l,l} = e^{j2\pi f_0 (N-l)}$ and $\mathbf{P}$ is defined as in (2).

(a) Show that for $\mathbf{Y} = \mathbf{F}\mathbf{y}$, $\mathbf{X} = \mathbf{F}\mathbf{x}$,

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \mathbf{Z} \tag{7}$$

and prove that

$$\mathbf{G} = \mathbf{F}\mathbf{S}\mathbf{F}^*\mathbf{D}.$$

(b) If $f_0 \neq 0$, we see from part (a) that $\mathbf{G}$ is no longer a diagonal matrix and therefore we do not obtain the parallel channel result of (4). We get inter-carrier interference (ICI), *i.e.,* we have

$$\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q \neq l} \mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \dots, N-1,$$

which shows that the other carriers interfere with $\mathbf{X}_l$. Compute the SINR (signal-to-interference plus noise ratio). Assume $\{\mathbf{X}_l\}$ are i.i.d, with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$. You can compute the SINR for the particular l and leave the expression in terms of $\{G(l,q)\}$.

(c) Find the filter $\mathbf{W}_l$, such that the MMSE criterion is fulfilled,

$$\min_{\mathbf{W}_l} \mathbb{E}|\mathbf{W}_l^*\mathbf{Y} - \mathbf{X}_l|^2.$$

You can again assume that $\{\mathbf{X}_l\}$ are i.i.d with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$ and that the receiver knows $\mathbf{G}$. You can now state the answer in terms of $\mathbf{G}$.

(d) Find an expression for $\mathbf{G}_{l,q}$ in terms of $f_0, N, \{d_l\}$.

Hint: Use the summation of the geometric series

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}.$$

Problem 4

In class we derived the Tomlinson-Harashima precoding for the real case (PAM constellation). In this problem, we will derive it for QAM (complex) constellation.
Let $C = \{(a, b) : a, b \in \{-1, +1\}\}$ be the vector mapping points in QAM constellation. We use the orthogonal bases to obtain the following waveform

$$x[k] = a[k]\phi_1(t - kT) + b[k]\phi_2(t - kT).$$

Consider the channel model $Y(D) = \| p \| Q(D)X(D) + Z(D)$, where $Q(D) = \gamma_0 F(D)F^*(D^{-*})$

(a) Find $B(D)$ and $W(D)$ for the ZF-DFE.

(b) We want to use Tomlinson-Harashima precoding for this model. Put the feedback filter at the transmitter part and show that there is no ISI in the received signal after the matched filter.

(c) Design the modulo function in the structure given in Figure 1 such that the transmitted energy is not boosted.
Hint: Can you construct a modulo function independently for the two components?

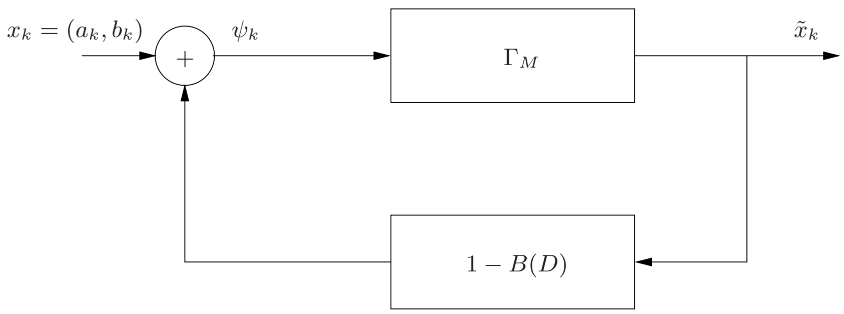

Figure 1: Tomlison-Harashima precoder