Consider the discrete time model studied in class

\[ Y(D) = ||p||Q(D)X(D) + Z(D), \]

where \( S_x(D) = E_x, \) \( S_z(D) = N_0Q(D) \) with \( Q(D) = Q^*(D^{-*}). \) In class we derived the MMSE-DFE, but in this problem we consider a slightly different structure shown in Figure 1. As in class we consider perfect decision feedback, i.e., all past decisions are correct. Let

\[
R(D) = H(D)Y(D),
R'(D) = R(D) + (1 - B(D))(X(D) - R(D)).
\]

We restrict \( B(D) \) to be causal and monic, i.e.,

\[ B(D) = 1 + \sum_{l=1}^{\infty} b_l D^l. \]

We choose \( H(D) \) and \( B(D) \) to minimize

\[ \mathbb{E} \left( |x_k - r'_k|^2 \right) \]

as we did in class.

(a) Find \( H(D) \) in terms of \( B(D) \) by using orthogonality principle.
(b) Set-up the prediction problem by proving that the error
\[ E(D) = X(D) - R'(D) = B(D)X(D) - B(D)H(D)Y(D). \]

Use the solution of \( H(D) \) in terms of \( B(D) \) found in part (a) to show that
\[ E(D) = B(D)U(D) \]
and find the expression for \( U(D) \).
Show that
\[ S_U(D) = \frac{N_0/||p||^2}{Q(D) + 1/SNR_{MF}}. \]
Given this can you comment on the values of \( H(D) \) and \( B(D) \) with respect to the quantities derived in class. In particular, is the noise-prediction DFE the same as the MMSE-DFE derived in the class?

(c) If \( B(D) = 1 \), what does the structure in Figure 1 become?

**Problem 2**

Consider the channel
\[ y(k) = ||p||x(k) * q(k) + z(k), \]
where \( q(k) = \delta(k) + b\delta(k - 1) + b\delta(k + 1) \) and \( z(k) \) is zero-mean Gaussian noise with power spectral density (PSD) \( S_z(D) = N_0 Q(D) \). Assume that
\[ b = \sqrt{\frac{N_0}{E_z||p||^2}} = \frac{1}{2}. \]
Remember that by definition, \( SNR_{MF} = \frac{E_z||p||^2}{N_0} \). In this problem, we consider a zero-forcing decision-feedback equalizer (ZF-DFE).

(a) Find the factorization
\[ Q(D) = \nu_0 P_c(D)P^*_c(D^{-*}), \]
where \( P_c(D) \) should be monic and causal.

(b) Find \( B(D) \) and \( W(D) \) for the ZF-DFE.