Problem 1

Consider transmission over an ISI channel and the channel after matched filtering is

\( Y(D) = p \| Q(D)X(D) + Z(D) \). 

Let \( \{q_k\} = \langle \tilde{\phi}_0, \tilde{\phi}_k \rangle \), is given as the following

\[
q_k = \begin{cases} 
2^{-\frac{|k|-1}{2}} & \text{if } k \text{ is odd,} \\
\frac{5}{3}2^{-\frac{|k|}{2}} & \text{if } k \text{ is even.}
\end{cases}
\]

and \( S_z(D) = N_0Q(D) \) and \( Q(D) \) is the \( D \)-transform of \( \{q_k\} \). Find the whitening filter \( W(D) \) to whiten the noise. Choose the whitening filter such that the resulting communication channel after the whitening filter is causal. That is, \( Q(D)W(D) \) is causal.

Problem 2

Consider transmission over an ISI channel with PAM constellation. Let us use

\( \phi(t) = \frac{1}{\sqrt{T}} \text{sinc} \left( \frac{t}{T} \right) \)

for symbol duration of \( T \) and the ISI channel is characterized as \( h(t) = \delta(t) - 0.4\delta(t-T) \) and assume that additive white Gaussian noise has power spectral density \( N_0 \).

(a) Determine the pulse response \( p(t) \).
(b) Find \( \| p \| \) and \( \tilde{\phi}(t) \).
(c) Find the autocorrelation function of the noise after sampling the output of the matched filter. Find the whitening filter such that the resulting channel is causal.
(d) Assume that \( N_0 = 0.49 \), size of PAM is 2, and \( x_i \in \{ \pm 1 \} \). Let the output of the whitened matched filter is \( \{0.7, -1.1, -0.2, 0.9, -0.6, 0.9\} \). Find the maximum likelihood sequence using the Viterbi algorithm. Assume the initial and last states are +1.
(e) Using the same assumptions as part (d), apply the BCJR algorithm to find the sequence which minimizes the symbol-to-symbol error. Compare the two results and comment.

Problem 3

In this exercise, we will show how the maximum a posteriori (MAP) detection rule can be extended to detection of sequences and also to risk-minimization for sequences.

Let \( X \) be a sequence of length \( N \), where every entry takes values in \( \{x_0, x_1\} \). Let \( Y \) be a random sequence (vector) of length \( N \), taking values in \( \mathbb{R}^N \). Assume that the statistics of \( X \) are given by \( P[ X = \hat{x} ] = \prod_{k=0}^{N-1} p(\hat{x}_k) \) for every possible sequence \( \hat{x} \in \{x_0, x_1\}^N \). The statistics of \( Y \) are given by some conditional density function \( P_{Y|X}(y|x) \) corresponding to an inter-symbol interference channel of memory \( \nu \), i.e., \( Y(k) = \sum_{i=0}^{\nu} f_ix(k-i) + Z(k) \).
(a) Show that by Bayes rule, the following identity holds
\[
P[X(n) = x_0 | Y = y] P_Y(y) = P_{Y|X(n)}(y|x_0) P[X(n) = x_0].
\]

(b) Read the proof of optimality of the MAP rule in Prof. Diggavi’s lecture notes. Show that one can carry out the same proof for detecting the symbol \(X(n)\) using the whole sequence \(y\). The resulting MAP rule should be
\[
\hat{x}(n) = \arg \max_{j \in \{0, 1\}} P[X(n) = x_j | Y = y].
\]

(c) Now we consider a setup with costs. Let \(C_{ij}\) be the cost of detecting \(x_i\) when the actual symbol was \(x_j\). Assume that \(C_{00} = C_{11} = 0\). Given that we are using the rule \(H^{(n)}\) for detecting \(X(n)\), the risk function is given by
\[
R(H^{(n)}) = C_{01} P[H^{(n)}(Y) = x_0 | X(n) = x_1] P[X(n) = x_1]
+ C_{10} P[H^{(n)}(Y) = x_1 | X(n) = x_0] P[X(n) = x_0].
\]
Show that the problem of minimizing this risk is equivalent to minimizing the probability of error of a setup without costs, with the modified prior probabilities:
\[
\tilde{P}[X(n) = x_1] = \frac{C_{01} P[X(n) = x_1]}{\alpha}
\]
\[
\tilde{P}[X(n) = x_0] = \frac{C_{10} P[X(n) = x_0]}{\alpha},
\]
where \(\alpha = C_{01} P[X(n) = x_1] + C_{10} P[X(n) = x_0]\).

(d) Use everything that you know so far to derive the risk-minimizing MAP rule for detecting \(X(n)\).

(e) Write down the basic steps of the BCJR algorithm for risk minimization.