Solutions to Homework 1

Problem 1
a. MAP rule for binary hypothesis testing:

\[
\frac{p_{Y|H}(y|H=0)}{p_{Y|H}(y|H=1)} \geq \frac{P_H(1)}{P_H(0)} = \frac{1}{1000}
\]

We have

\[
l(y) = \frac{p_{Y|H}(y|0)}{p_{Y|H}(y|1)} = \begin{cases} 
\infty & y \leq 0 \\
\frac{1-y}{y} & 0 < y < 1 \\
0 & 1 \leq y 
\end{cases}
\]

Solving for \( l(t) = 1/999 \) gives \( t = 999/1000 \).

b. The probabilities of false alarm and miss are given by \( p_f = P_h(0)p_{Y|H}(Y \geq t|0) \) and \( p_m = P_H(1)p_{Y|H}(Y < t|1) \) respectively. Obviously the threshold of the decision scheme should be between 0 and 1 (why?). The expected cost is \( 200p_f + 10000p_m \).

We have

\[
p_{Y|H}(Y \geq t|0) = \int_t^1 (1-y)dy = \frac{(1-t)^2}{2}
\]

\[
p_{Y|H}(Y < t|1) = \int_0^t ydy = \frac{t^2}{2}
\]

Plugging in the values and minimizing the expected cost with respect to \( t \) gives \( t = 999/1049 \).

Problem 2
a. (i) Given only \( Y_1, Y_3 \) is not relevant. This is intuitive because \( Y_3 \) is a more noisier
version of $Y_1$. Mathematically,
\[
\begin{align*}
\frac{P_{X|Y_1,Y_3}(1 \mid y_1,y_3)}{P_{X|Y_1,Y_3}(0 \mid y_1,y_3)} &= \frac{P_X(1)P_{Y_1,Y_3|X}(y_1,y_3 \mid 1)}{P_X(0)P_{Y_1,Y_3|X}(y_1,y_3 \mid 0)} \\
&= \frac{P_X(1)P_{Y_1|X}(y_1 \mid 1)P_{Y_3|Y_1,X}(y_3 \mid y_1,1)}{P_X(0)P_{Y_1|X}(y_1 \mid 0)P_{Y_3|Y_1,X}(y_3 \mid y_1,0)} \\
&= \frac{P_X(1)P_{Y_1|X}(y_1 \mid 1)P_{Y_3|Y_1}(y_3 \mid y_1)}{P_X(0)P_{Y_1|X}(y_1 \mid 0)P_{Y_3|Y_1}(y_3 \mid y_1)} \\
&= \frac{P_{X|Y_1}(1 \mid y_1)}{P_{X|Y_1}(0 \mid y_1)}
\end{align*}
\]

The equality (a) is due to the fact that given $Y_1$, $Y_3$ is independent of $X$ (it is clear if you write $Y_3 = Y_1 + N_2$).

(ii) Given both $Y_1$ and $Y_2$, $Y_3$ is relevant. This is also intuitive because given only $Y_1, Y_2$ we can estimate $X$ with some probability of error. But given all three, we can estimate it correctly, simply by adding all three of them ($Y_1 + Y_2 + Y_3 = X$). The result can be proven more formally as above, but in this case we have to show that
\[
\frac{P_{X|Y_1,Y_2}(1 \mid y_1,y_2)}{P_{X|Y_1,Y_2}(0 \mid y_1,y_2)} \neq \frac{P_{X|Y_1,Y_3}(1 \mid y_1,y_2,y_3)}{P_{X|Y_1,Y_3}(0 \mid y_1,y_2,y_3)}
\]

b. (i) Yes, given only $Y_1$, $Y_2$ is relevant. Because they are both independent observations and having more observations will decrease the probability of error.

(ii) Yes, given only $Y_1$, $Y_3$ is relevant. Because $Y_3$ gives some knowledge about $N_1$.

(iii) No, given both $Y_1$ and $Y_2$, $Y_3$ is not relevant. Because $Y_3 = Y_1 - Y_2$.

**Problem 3**

The 3 signals are $\cos(t)$, $\cos(t + \pi/3)$ and $\sin(t)$. The basis can be taken as $\cos(t)$ and $\sin(t)$, because these two are orthogonal and $\cos(t + \pi/3)$ can be written as a linear combination of these two signals.

**Problem 4**

a. Clearly here the bandwith ($B$) is $B_2$. The sampling theorem states that samples should be separated by $T_s = 1/2B$ for being able to reconstruct the original signal $\in L^2$. The sampling frequency $F_s$ is then $2B_2$.

We can do better by observing that a portion of width $2B_1$ of the spectrum is not used. Furthermore, since the spectrum is symmetric, all the information about the function $s(t)$ is contained in the “positive half” of the spectrum. Hence, we can reduce the sampling frequency by shifting the spectrum towards the center.

- Remove negative part of the spectrum
Define the filter with impulse response $h_>(t)$ via its Fourier transform $H_>(f)$

$$H_>(f) = \begin{cases} 
1 & \text{iff } f > 0 \\
1/2 & \text{iff } f = 0 \\
0 & \text{iff } f < 0
\end{cases}$$

or equivalently $H_>(f) = \frac{1}{2} + \frac{1}{2}\text{sgn}(f)$. This is a filter that removes the negative portion of the spectrum. The analytic equivalent of $s(t)$ is given by

$$S_A = \sqrt{2}S(f)H_>(f)$$

where the factor $\sqrt{2}$ guarantees that $s(t)$ and $s_A(t)$ have the same norm.

**Baseband Signal**

The baseband equivalent of $s(t)$ is defined as:

$$s_E(t) = s_A(t)e^{-j2\pi \frac{B_1+B_2}{2}t}$$

$$S_E(f) = S_A(f + \frac{B_1+B_2}{2}).$$

**Sampling and reconstruction**

From that point on the signal will occupy the band $[-\frac{B_2-B_1}{2}, \frac{B_2-B_1}{2}]$. The sampling frequency is $B_2 - B_1$ and $T_s = \frac{1}{B_2-B_1}$. The original signal is immediately obtained from $s_E(t)$ by

$$s(t) = \sqrt{2}\Re\{s_E(t)e^{j2\pi \frac{B_1+B_2}{2}t}\}$$

where $\Re$ denotes the real part.

b. We want to know whether $\{q(t-kT)\}_{k=-\infty}^{+\infty}$ is an orthonormal set of signals. Nyquist criterion says that the answer is in the affirmative if

$$\sum_{k=-\infty}^{+\infty} |Q(f + \frac{k}{T})|^2 = T \text{ for } f \in [-\frac{1}{2T}, \frac{1}{2T}]$$.

Since $q(t)$ is real, $|Q(f)|$ is symmetric and therefore we have

$$|Q(f)|^2 = \begin{cases} 
T - T^2f & |f| \leq 1/T \\
0 & \text{else}
\end{cases}.$$ 

Overlaying $\{q(t-kT)\}_{k=-\infty}^{+\infty}$ in the graph below, it is seen that $q(t)$ is a Nyquist pulse.
Problem 5 Let
\[
p(x, y) = \begin{cases} 
\pi^{-1} e^{-\frac{1}{2}(x^2 + y^2)} & xy > 0 \\
0 & \text{else}
\end{cases}
\]
be the joint density of random variables \(X\) and \(Y\). Then \(X\) and \(Y\) are identically distributed with density
\[
p(x) = (2\pi)^{-1/2} e^{-\frac{1}{2}x^2},
\]
and thus they are individually Gaussian. However, clearly the pair \(X, Y\) is not Gaussian.

Problem 6 \(\hat{U} = \begin{bmatrix} U_R \\
U_I \end{bmatrix}\), and thus \(K_U = \begin{bmatrix} K_{U_R U_R} & K_{U_R U_I} \\
K_{U_I U_R} & K_{U_I U_I} \end{bmatrix}\). On the other hand
\[
K_U = (K_{U_R U_R} + K_{U_I U_I}) + j(K_{U_I U_R} - K_{U_R U_I})
\]
and
\[
0 = J_U = (K_{U_R U_R} - K_{U_I U_I}) + j(K_{U_I U_R} + K_{U_R U_I}).
\]

Thus, \(K_U = 2K_{U_R U_R} + j2K_{U_I U_R}\) and \(K_{\hat{U}} = \begin{bmatrix} K_{U_R U_R} & -K_{U_I U_R} \\
K_{U_I U_R} & K_{U_R U_R} \end{bmatrix}\). So we see that \(\hat{K}_U = 2K_U\).

Problem 7
(a) Let \(x_E(t) = \alpha(t) \exp[j\beta(t)]\). Then
\[
x(t) = \sqrt{2} \Re \{x_E(t) \exp[j2\pi f_0 t]\} = \sqrt{2} \Re \{\alpha(t) \exp[j\beta(t)] \exp[j2\pi f_0 t]\} = \sqrt{2} \Re \{\alpha(t) \exp[j(2\pi f_0 t + \beta(t))])\} = \sqrt{2} \alpha(t) \cos[2\pi f_0 t + \beta(t)].\]
We thus have

\[ a(t) = \sqrt{2}a(t) = \sqrt{2}||x_E(t)|| \]

and

\[ \theta(t) = \beta(t) = \tan^{-1} \frac{\Im\{x_E(t)\}}{\Re\{x_E(t)\}}. \]

This shows that a bandpass signal is one that is modulated both in amplitude and in phase.

(b) Let \( x_E(t) = x_R(t) + jx_I(t) \). Then

\[
x(t) = \sqrt{2}\Re\{x_E(t) \exp[j2\pi f_0 t]\}
\]

\[
= \sqrt{2}\Re\{[x_R(t) + jx_I(t)] \exp[j2\pi f_0 t]\}
\]

\[
= \sqrt{2}[x_R(t) \cos(2\pi f_0 t) - x_I(t) \sin(2\pi f_0 t)].
\]

Hence we have

\[ x_{EI}(t) = \sqrt{2}\Re\{x_E(t)\} \]

and

\[ x_{EQ}(t) = \sqrt{2}\Im\{x_E(t)\}. \]

(c) We guess that

\[ x_E(t) = \frac{A(t)}{\sqrt{2}} \exp(j\varphi). \]

Indeed

\[
x(t) = \sqrt{2}\Re\{x_E(t) \exp(j2\pi f_0 t)\}
\]

\[
= \sqrt{2}\Re\{\frac{A(t)}{\sqrt{2}} \exp(j\varphi) \exp(j2\pi f_0 t)\}
\]

\[
= \Re\{A(t) \exp[j(2\pi f_0 t + \varphi)]\}
\]

\[
= A(t) \cos(2\pi f_0 t + \varphi).
\]