Solutions to Homework 1

Problem 1

a. MAP rule for binary hypothesis testing:

$$\frac{p_{Y|H}(y|0)}{p_{Y|H}(y|1)} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{P_H(1)}{P_H(0)} = \frac{1}{999}$$

We have

$$l(y) = \frac{p_{Y|H}(y|0)}{p_{Y|H}(y|1)} = \begin{cases} \infty & y \leq 0 \\ \frac{1-y}{y} & 0 < y < 1 \\ 0 & 1 \leq y \end{cases}.$$

Solving for $l(t) = 1/999$ gives $t = 999/1000$.

b. The probabilities of false alarm and miss are given by $p_f = P_h(0)p_{Y|H}(Y \geq t|0)$ and $p_m = P_H(1)p_{Y|H}(Y < t|1)$ respectively. Obviously the threshold of the decision scheme should be between 0 and 1 (why?). The expected cost is $200p_f + 10000p_m$. We have

$$p_{Y|H}(Y \geq t|0) = \int_t^1 (1-y)dy = \frac{(1-t)^2}{2}$$
$$p_{Y|H}(Y < t|1) = \int_0^t y dy = \frac{t^2}{2}$$

Plugging in the values and minimizing the expected cost with respect to t gives $t = 999/1049$.

Problem 2

a. (i) Given only Y_1, Y_3 is not relevant. This is intuitive because Y_3 is a more noisier version of Y_1. Mathematically,

$$\begin{aligned}\frac{P_{X|Y_1,Y_3}(1 \mid y_1, y_3)}{P_{X|Y_1,Y_3}(0 \mid y_1, y_3)} &= \frac{P_X(1)P_{Y_1,Y_3|X}(y_1, y_3 \mid 1)}{P_X(0)P_{Y_1,Y_3|X}(y_1, y_3 \mid 0)} \\ &= \frac{P_X(1)P_{Y_1|X}(y_1 \mid 1)P_{Y_3|Y_1,X}(y_3 \mid y_1, 1)}{P_X(0)P_{Y_1|X}(y_1 \mid 0)P_{Y_3|Y_1,X}(y_3 \mid y_1, 0)} \\ &\overset{(a)}{=} \frac{P_X(1)P_{Y_1|X}(y_1 \mid 1)P_{Y_3|Y_1}(y_3 \mid y_1)}{P_X(0)P_{Y_1|X}(y_1 \mid 0)P_{Y_3|Y_1}(y_3 \mid y_1)} \\ &= \frac{P_{X|Y_1}(1 \mid y_1)}{P_{X|Y_1}(0 \mid y_1)}\end{aligned}$$

The equality (a) is due to the fact that given Y_1, Y_3 is independent of X (it is clear if you write $Y_3 = Y_1 + N_2$).
(ii)Given both Y_1 and Y_2, Y_3 is relevant. This is also intuitive because given only Y_1, Y_2 we can estimate X with some probability of error. But given all three, we can estimate it correctly, simply by adding all three of them ($Y_1 + Y_2 + Y_3 = X$). The result can be proven more formally as above, but in this case we have to show that

$$\frac{P_{X|Y_1,Y_2}(1 \mid y_1, y_2)}{P_{X|Y_1,Y_2}(0 \mid y_1, y_2)} \neq \frac{P_{X|Y_1,Y_2,Y_3}(1 \mid y_1, y_2, y_3)}{P_{X|Y_1,Y_2,Y_3}(0 \mid y_1, y_2, y_3)}$$

b. (i) Yes, given only Y_1, Y_2 is relevant. Because they are both independent observations and having more observations will decrease the probability of error.
(ii) Yes, given only Y_1, Y_3 is relevant. Because Y_3 gives some knowledge about N_1.
(iii) No, given both Y_1 and Y_2, Y_3 is not relevant. Because $Y_3 = Y_1 - Y_2$.

Problem 3

The 3 signals are $\cos(t)$, $\cos(t + \pi/3)$ and $\sin(t)$. The basis can be taken as $\cos(t)$ and $\sin(t)$, because these two are orthogonal and $\cos(t + \pi/3)$ can be written as a linear combination of these two signals.

Problem 4

a. Clearly here the bandwith (B) is B_2. The sampling theorem states that samples should be seperated by $T_s = 1/2B$ for being able to reconstruct the original signal $\in \mathcal{L}^2$. The sampling frequency F_s is then $2B_2$.
We can do better by observing that a portion of width $2B_1$ of the spectrum is not used. Furthermore, since the spectrum is symmetric, all the information about the function $s(t)$ is contained in the "positive half" of the spectrum. Hence, we can reduce the sampling frequency by shifting the spectrum towards the center.

- Remove negative part of the spectrum

Define the filter with impulse response $h_>(t)$ via its Fourier transform $H_>(f)$

$$H_>(f) = \begin{cases} 1 & \text{if} f > 0 \\ 1/2 & \text{if} f = 0 \\ 0 & \text{if} f < 0 \end{cases}$$

or equivalently $H_>(f) = \frac{1}{2} + \frac{1}{2}\text{sgn}(f)$. This is a filter that removes the negative portion of the spectrum. The analytic equivalent of $s(t)$ is given by

$$S_A = \sqrt{2}S(f)H_>(f)$$

where the factor $\sqrt{2}$ guarantees that $s(t)$ and $s_A(t)$ have the same norm.

- Baseband Signal
  The baseband equivalent of $s(t)$ is defined as:

$$s_E(t) = s_A(t)e^{-j2\pi\frac{B_1+B_2}{2}t}$$
$$S_E(f) = S_A(f + \frac{B_1 + B_2}{2}).$$

- Sampling and reconstruction
  From that point on the signal will occupy the band $[-\frac{B_2-B_1}{2}, \frac{B_2-B_1}{2}]$. The sampling frequency is $B_2 - B_1$ and $T_s = \frac{1}{B_2-B_1}$. The original signal is immediately obtained from $s_E(t)$ by

$$s(t) = \sqrt{2}\Re\{s_E(t)e^{j2\pi\frac{B_1+B_2}{2}t}\}$$

  where $\Re$ denotes the real part.

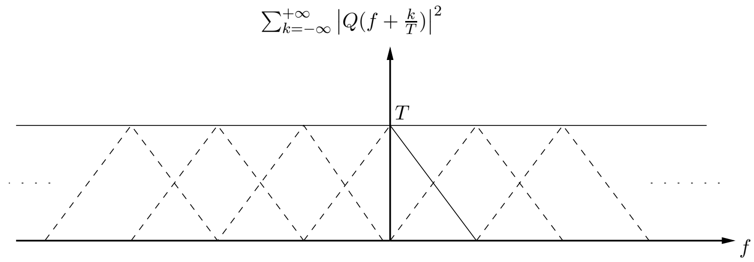
b. We want to know whether $\{q(t-kT)\}_{k=-\infty}^{+\infty}$ is an orthonormal set of signals. Nyquist criterion says that the answer is in the affirmative if

$$\sum_{k=-\infty}^{+\infty} \left|Q(f + \frac{k}{T})\right|^2 = T \text{ for } f \in [-\frac{1}{2T}, \frac{1}{2T}].$$

Since $q(t)$ is real, $|Q(f)|$ is symmetric and therefore we have

$$|Q(f)|^2 = \begin{cases} T - T^2 f & |f| \leq 1/T \\ 0 & else \end{cases}.$$

Overlaying $\{q(t-kT)\}_{k=-\infty}^{+\infty}$ in the graph below, it is seen that $q(t)$ is a Nyquist pulse.



Problem 5 Let

$$p(x,y) = \begin{cases} \pi^{-1} x^{-\frac{1}{2}(x^2+y^2)} & xy > 0 \\ 0 & else \end{cases}$$

be the joint density of random variables X and Y. then X and Y are identically distributed with density

$$p(x) = (2\pi)^{-1/2} e^{-\frac{1}{2}x^2},$$

and thus they are individually Gaussian. However, clearly the pair X, Y is not Gaussian.

Problem 6 $\hat{\mathbf{U}} = \begin{bmatrix} \mathbf{U}_R \\ \mathbf{U}_I \end{bmatrix}$, and thus $K_{\hat{\mathbf{U}}} = \begin{bmatrix} K_{\mathbf{U}_R\mathbf{U}_R} & K_{\mathbf{U}_R\mathbf{U}_I} \\ K_{\mathbf{U}_I\mathbf{U}_R} & K_{\mathbf{U}_I\mathbf{U}_I} \end{bmatrix}$. On the other hand

$$K_{\mathbf{U}} = (K_{\mathbf{U}_R\mathbf{U}_R} + K_{\mathbf{U}_I\mathbf{U}_I}) + j(K_{\mathbf{U}_I\mathbf{U}_R} - K_{\mathbf{U}_R\mathbf{U}_I})$$

and

$$0 = J_{\mathbf{U}} = (K_{\mathbf{U}_R\mathbf{U}_R} - K_{\mathbf{U}_I\mathbf{U}_I}) + j(K_{\mathbf{U}_I\mathbf{U}_R} + K_{\mathbf{U}_R\mathbf{U}_I}).$$

Thus, $K_{\mathbf{U}} = 2K_{\mathbf{U}_R\mathbf{U}_R} + j2K_{\mathbf{U}_I\mathbf{U}_R}$ and $K_{\hat{\mathbf{U}}} = \begin{bmatrix} K_{\mathbf{U}_R\mathbf{U}_R} & -K_{\mathbf{U}_I\mathbf{U}_R} \\ K_{\mathbf{U}_I\mathbf{U}_R} & K_{\mathbf{U}_R\mathbf{U}_R} \end{bmatrix}$. So we see that $\widehat{K_{\mathbf{U}}} = 2K_{\hat{\mathbf{U}}}$.

Problem 7
(a) Let $x_E(t) = \alpha(t)\exp[j\beta(t)]$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}\Re\{\alpha(t)\exp[j\beta(t)]\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}\Re\{\alpha(t)\exp[j(2\pi f_0 t + \beta(t))]\} \\ &= \sqrt{2}\alpha(t)\cos[2\pi f_0 t + \beta(t)]. \end{aligned}$$

We thus have

$$a(t) = \sqrt{2}\alpha(t) = \sqrt{2}||x_E(t)||$$

and

$$\theta(t) = \beta(t) = \tan^{-1} \frac{\Im\{x_E(t)\}}{\Re\{x_E(t)\}}.$$

This shows that a bandpass signal is one that is modulated both in amplitude and in phase.

(b) Let $x_E(t) = x_R(t) + jx_I(t)$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}\Re\{[x_R(t) + jx_I(t)]\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}[x_R(t)\cos(2\pi f_0 t) - x_I(t)\sin(2\pi f_0 t)]. \end{aligned}$$

Hence we have

$$x_{EI}(t) = \sqrt{2}\Re\{x_E(t)\}$$

and

$$x_{EQ}(t) = \sqrt{2}\Im\{x_E(t)\}.$$

(c) We guess that

$$x_E(t) = \frac{A(t)}{\sqrt{2}}\exp(j\varphi).$$

Indeed

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)\exp(j2\pi f_0 t)\} \\ &= \sqrt{2}\Re\{\frac{A(t)}{\sqrt{2}}\exp(j\varphi)\exp(j2\pi f_0 t)\} \\ &= \Re\{A(t)\exp[j(2\pi f_0 t + \varphi)]\} \\ &= A(t)\cos(2\pi f_0 t + \varphi). \end{aligned}$$