MIDTERM

Wednesday December 14, 2005, 9:00-13:00
This exam has 6 problems and 80 points in total (+ 10 Bonus points).

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- We will use the following notation: if $W(D)$ is the D-transform of w_k, we will denote w_k by

$$w_k = \mathcal{D}^{-1}\left[W(D)\right]\Big|_k .$$

- The Fourier transform of $a(t) = e^{-\alpha|t|}$ is

$$A(f) = \frac{\frac{2}{\alpha}}{(1+\frac{1}{\alpha}j2\pi f)(1-\frac{1}{\alpha}j2\pi f)}.$$

Good Luck!

Problem 1

[THE Z-CHANNEL **(12pts)**]
Consider the following binary channel shown in Figure 1.

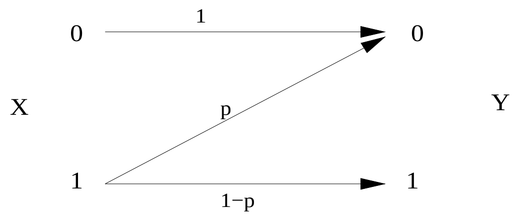

Figure 1: The Z-channel.

[*3pts*] **(a)** Let us consider a single transmission with $\mathbb{P}[X=0]=q$, $\mathbb{P}[X=1]=1-q$. Assume that the prior q is known at the decoder. Given observation Y, give the decision rule that minimizes the probability of error, i.e., $\mathbb{P}\left[\hat{X}\neq X\right]$.
You can assume that $q\in[0,\frac{1}{2}]$ and $p\in[0,\frac{1}{2}]$.
Hint: Consider cases when $\frac{q}{1-q}\lesseqgtr p$.

[*3pts*] **(b)** Now, consider successive transmissions where the **same** input symbol X is transmitted n times over independent realizations of this channel. As before, consider the priors to be $\mathbb{P}[X=0]=q$, $\mathbb{P}[X=1]=1-q$, again known at the decoder. Given the observations $Y_1,\ldots,Y_n$, give the decision rule that minimizes the error probability.

[*6pts*] **(c)** Now, let $\mathbf{X}$ be a vector that has the following two hypotheses:

$$\mathbf{X}=\begin{cases}\mathbf{1} & \text{w.p. } \frac{1}{2} \implies \text{hypothesis } H_0\\ \mathbf{S} & \text{w.p. } \frac{1}{2} \implies \text{hypothesis } H_1\end{cases},$$

where $\mathbf{1}=(1,\ldots,1)$, and $\mathbf{S}=(S_1,\ldots,S_n)$ is an i.i.d. process with

$$\mathbb{P}[S_1=s_1]=\begin{cases}\frac{1}{2} & \text{for } s_1=0\\ \frac{1}{2} & \text{for } s_1=1\end{cases}.$$

Given n observations $Y_1,\ldots,Y_n$, we want to decide between the two hypotheses. Find the maximum likelihood (ML) rule to decide between these two hypotheses, i.e., hypothesis H_0 or H_1.
Hint: Write out the distributions of $(Y_1,\ldots,Y_n)$ under the two hypotheses.

Problem 2

[EXPONENTIAL DETECTION **(10pts + 7pts Bonus)**]
Consider the following channel:

$$Y=X+Z,$$

where Z is independent of X and the probability density function of Z is

$$p_Z(z)=\frac{1}{2}e^{-|z|}.$$

[*5pts*] **(a)** If $\mathbb{P}[X=-1]=q$, $\mathbb{P}[X=1]=1-q$, find the decision regions of the MAP decoder for this channel.

[*5pts*] **(b)** If the prior q is unknown, find the minmax detection rule $\min_H \max_q P_{e,H}(q)$ for this channel. Please do not just state the decision rule, but prove the result.
Hint: You can use a relation between $\mathbb{P}[\text{error}|X=1]$ and $\mathbb{P}[\text{error}|X=-1]$.

[*Bonus: 7pts*] **(c)** Now assume that we make two observations

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2, \end{aligned}$$

where Z_1 and Z_2 are independent and identically distributed with densities

$$p_{Z_1}(z) = p_{Z_2}(z) = \frac{1}{2}e^{-|z|}.$$

Let $\mathbb{P}[X=-1]=\mathbb{P}[X=1]=\frac{1}{2}$. Prove that the optimal decision rule for this channel is given by the decision regions in Figure 2.

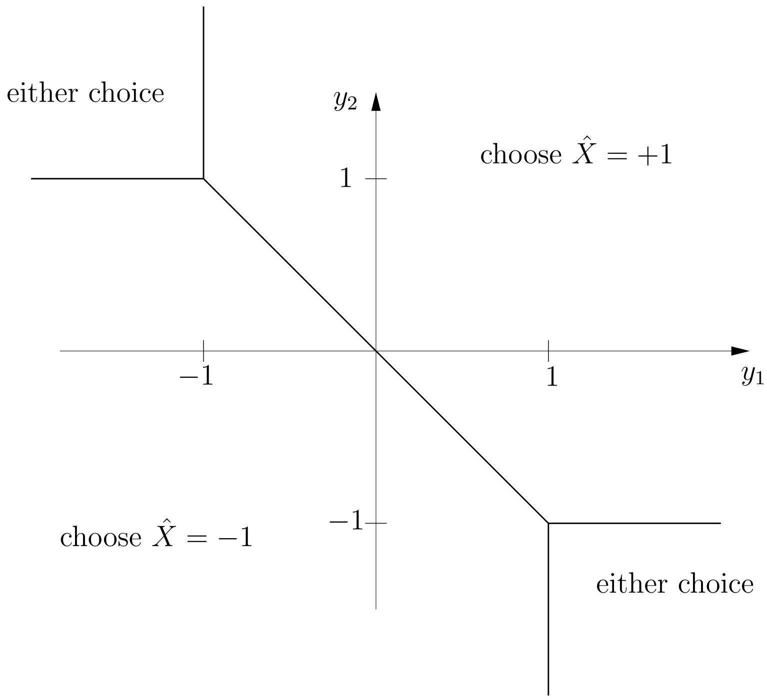

Figure 2: Decision regions.

Problem 3

[Colored Passband Process **(14pts)**]
Suppose we have access to a real passband process

$$x(t) = x_I(t)\cos(\omega_c t) - x_Q(t)\sin(\omega_c t)$$

where $x_I(t)$ and $x_Q(t)$ are zero-mean jointly Gaussian random processes, with

$$\mathbb{E}[x_I(t)x_I(t-\tau)] = e^{-2|\tau|} = \mathbb{E}[x_Q(t)x_Q(t-\tau)].$$

Further, let $x_I(t)$ and $x_Q(t)$ be independent.

[*8pts*] **(a)** Suppose we want to produce a baseband process $y_{bb}(t)$ such that

$$y_{bb}(t) = y_I(t) + jy_Q(t)$$

with

$$\mathbb{E}[y_I(t)y_I(t-\tau)] = \frac{1}{2}\delta(\tau) = \mathbb{E}[y_Q(t)y_Q(t-\tau)].$$

and $y_I(t)$ and $y_Q(t)$ are zero-mean random processes. Show how to produce such a baseband process using $x(t)$.

[*6pts*] **(b)** Now if we want to produce a passband process $z(t)$ such that

$$z(t) = z_I(t)\cos(\tilde{\omega}_c t) - z_Q(t)\sin(\tilde{\omega}_c t)$$

with a **given** $\tilde{\omega}_c > \omega_c$. Furthermore $z_I(t)$ and $z_Q(t)$ to be zero-mean independent Gaussian processes with auto-correlation functions

$$\mathbb{E}[z_I(t)z_I(t-\tau)] = e^{-3|\tau|} = \mathbb{E}[z_Q(t)z_Q(t-\tau)].$$

Starting from $x(t)$, show how we can produce the desired $z(t)$.

Hint: For both parts (a) and (b), you can give a procedure in the frequency domain.

Problem 4

[LINEAR PREDICTION **(13pts)**]
Given observation

$$y_k = x_k + z_k$$

where $\{z_k\}$ is an iid Gaussian random process with zero-mean and unit variance which is independent of x_k. Let $\{x_k\}$ be zero-mean with $\mathbb{E}[x_k x_{k-\ell}] = r_x(\ell)$.

[*9pts*] **(a)** We want to produce $\hat{x}_k$, given observation $\{y_\ell\}_{\ell=-\infty}^{k-1}$ using a linear predictor

$$\hat{x}_k = \sum_{n=1}^{\infty} a_n y_{k-n}.$$

Find an expression for the best MMSE linear predictor for $\hat{x}_k$, *i.e.,* assuming the Paley-Wiener condition holds for all the spectra involved. That is, find $\{a_n\}_{n=1}^{\infty}$ in terms (of a function) of $S_x(D)$.

[*4pts*] **(b)** If $r_x(\ell) = e^{-2|\ell|}$, where l is an integer, find an explicit expression for the linear predictor.

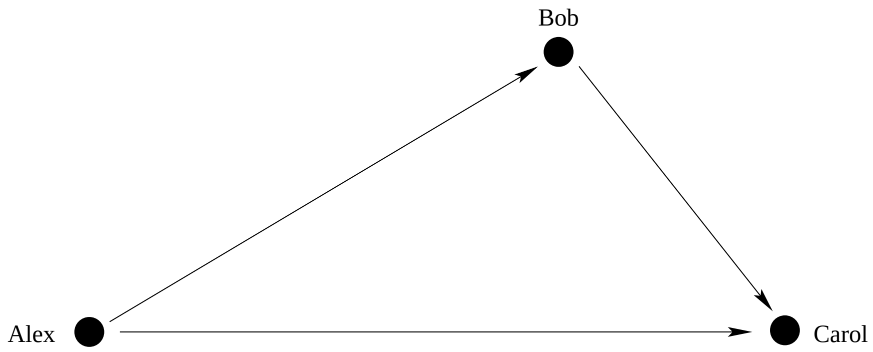

Figure 3: Relay estimation

Problem 5

[RELAY ESTIMATION **(15pts)**]
Suppose Alex had a wireless transmitter in the INR building and wants to communicate with Carol in the BC building. However, Alex has a friend Bob who is in the PSE building and is willing to help Alex with his mission. Both Carol and Bob have additive Gaussian noise channels.

$$\begin{aligned} y_B(t) &= x_A(t) + z_B(t) \\ y_C(t) &= x_A(t) + x_B(t) + z_C(t) \end{aligned}$$

where $z_B(t)$ and $z_C(t)$ are independent zero-mean white Gaussian processes with power spectral density N_0. Now, Alex sends out the sequence $\{x[n]\}$ using a basis function $\phi(t)$

$$x_A(t) = \sum_n x[n]\phi(t - nT),$$

where $x[n]$ is an i.i.d. sequence with variance $\mathcal{E}_x$. Bob after listening to Alex's message uses a matched filter to collect sufficient statistics and obtains

$$y_B[k] = x_A[k] + z_B[k]. \tag{1}$$

He then forwards the symbol $y[k]$ at the next transmission and therefore sends

$$x_B(t) = \sum_k y_B[k]\phi(t - (k+1)T)$$

as the transmitted waveform (see Figure 3).
You can assume the shifted versions of the basis are orthonormal

$$< \phi_0, \phi_k >= \int \phi^*(t)\phi(t - kT)dt = \delta_k = \int \phi^*(t - (n-k)T)\phi(t - nT)dt.$$

[*4pts*] **(a)** Find the spectrum $S_{z_B}(D)$ of $z_B[k]$ in (1) after the matched filter operation by Bob.

[*7pts*] **(b)** Now Carol receives the superposition of the signal from Alex and Bob, and uses a matched filter receiver to the received signal to obtain a set of sufficient statistics of the input sequence $x[k]$. Express the output of the matched filter receiver, $y_C[k]$, in terms of the crosscorrelation of the shifted basis function.

[*4pts*] **(c)** Now, Carol wants to use a linear filter $w[k]$ to estimate $x[k]$, which minimizes the estimation error

$$\mathbb{E}\Big|x[k] - w[k] * y_C[k]\Big|^2$$

between the estimate and the signal sent by Alex. Find the optimal filter $w(k)$.

Problem 6

[PERFORMANCE OF EQUALIZERS **(16pts + 3pts Bonus)**]
Recall the discrete-time channel after matched filtering and sampling:

$$Y(D) = ||p||Q(D)X(D) + Z(D).$$

Recall also the definition

$$SNR_{MFB} = \frac{\mathcal{E}_x ||p||^2}{N_0}.$$

[*5pts*] **(a)** We proved in class that the MMSE-DFE uses the spectral factorization

$$Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 G(D) G^*(D^{-*}), \tag{2}$$

where $G(D)$ is causal, strictly stable and monic.
Remember that since $q_k = \langle \tilde{\varphi}_0, \tilde{\varphi}_k \rangle$, we can conclude that $q_0 = 1$.
Using this, prove that

$$1 + \frac{1}{SNR_{MFB}} = \gamma_0 ||g||^2,$$

where

$$||g||^2 = \sum_{n=0}^{\infty} |g_n|^2.$$

[*3pts*] **(b)** Now, show that for any $G(D)$ satisfying equation (2), $||g||^2 \geq 1$, with equality if and only if $Q(D) = 1$.
Therefore using this we have

$$\gamma_0 \leq 1 + \frac{1}{SNR_{MFB}}.$$

[*5pts*] **(c)** Now, the MMSE-LE is given by (as shown in class)

$$W_{MMSE-LE}(D) = \frac{1}{||p|| \left(Q(D) + \frac{1}{SNR_{MFB}} \right)}.$$

We also showed in class that for the MMSE-LE, the error spectrum is

$$S_{EE}^{MMSE-LE}(D) = \frac{N_0}{||p||^2} \frac{1}{Q(D) + \frac{1}{SNR_{MFB}}} = \frac{N_0}{||p||} W_{MMSE-LE}(D).$$

Hence

$$\sigma^2_{MMSE-LE} = \frac{N_0}{||p||} w_{MMSE-LE}[0] = \frac{N_0}{||p||^2} \mathcal{D}^{-1} \left[\frac{1}{Q(D) + \frac{1}{SNR_{MFB}}} \right] \Bigg|_0$$

(see the instructions page for the notation). Now,

$$Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 G(D) G^*(D^{-*})$$

and

$$\frac{1}{Q(D) + \frac{1}{SNR_{MFB}}} = \beta L(D) L^*(D^{-*}),$$

where $\beta = \frac{1}{\gamma_0}$ and $L(D) = \frac{1}{G(D)}$.
Using this, prove that

$$\mathcal{D}^{-1}\left[\frac{1}{Q(D) + \frac{1}{SNR_{MFB}}}\right]\Bigg|_0 \geq \frac{1}{\gamma_0}. \tag{3}$$

[*3pts*] **(d)** We have shown in class that for the MMSE-DFE,

$$S_{EE}^{MMSE-DFE}(D) = \frac{N_0}{\gamma_0||p||^2}.$$

Hence

$$\sigma^2_{MMSE-DFE} = \frac{N_0}{\gamma_0||p||^2}.$$

Using this and inequality (3) show that

$$SNR_{MMSE-LE,U} \leq SNR_{MMSE-DFE,U},$$

with equality iff $Q(D) = 1$.

[*Bonus: 3pts*] **(e)** Use results from (b) and (c) to show that

$$SNR_{MMSE-LE,U} \leq SNR_{MMSE-DFE,U} \leq SNR_{MFB}.$$