MIDTERM
Wednesday December 14, 2005, 9:00-13:00
This exam has 6 problems and 80 points in total (+ 10 Bonus points).

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.

- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.

- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.

- We will use the following notation: if $W(D)$ is the $D$-transform of $w_k$, we will denote $w_k$ by
  \[ w_k = D^{-1}\left[ W(D) \right]_{k}. \]

- The Fourier transform of $a(t) = e^{-\alpha |t|}$ is
  \[ A(f) = \frac{2}{\alpha} \frac{j2\pi f}{(1 + \frac{1}{\alpha}j2\pi f)(1 - \frac{1}{\alpha}j2\pi f)}. \]

Good Luck!
Problem 1

[The Z-Channel (12pts)]
Consider the following binary channel shown in Figure 1.

![Z-channel diagram](image)

Figure 1: The Z-channel.

(a) Let us consider a single transmission with \( P[X = 0] = q, \ P[X = 1] = 1 - q \). Assume that the prior \( q \) is known at the decoder. Given observation \( Y \), give the decision rule that minimizes the probability of error, i.e., \( \mathbb{P}[\hat{X} \neq X] \).

You can assume that \( q \in [0, \frac{1}{2}] \) and \( p \in [0, \frac{1}{2}] \).

**Hint:** Consider cases when \( \frac{1}{1-q} \leq p \).

(b) Now, consider successive transmissions where the same input symbol \( X \) is transmitted \( n \) times over independent realizations of this channel. As before, consider the priors to be \( P[X = 0] = q, \ P[X = 1] = 1 - q \), again known at the decoder. Given the observations \( Y_1, \ldots, Y_n \), give the decision rule that minimizes the error probability.

(c) Now, let \( X \) be a vector that has the following two hypotheses:

\[
X = \begin{cases} 
1 & \text{w.p. } \frac{1}{2} \Rightarrow \text{hypothesis } H_0 \\
S & \text{w.p. } \frac{1}{2} \Rightarrow \text{hypothesis } H_1 
\end{cases}
\]

where \( 1 = (1, \ldots, 1) \), and \( S = (S_1, \ldots, S_n) \) is an i.i.d. process with

\[
P[S_1 = s_1] = \begin{cases} 
\frac{1}{2} & \text{for } s_1 = 0 \\
\frac{1}{2} & \text{for } s_1 = 1 
\end{cases}
\]

Given \( n \) observations \( Y_1, \ldots, Y_n \), we want to decide between the two hypotheses. Find the maximum likelihood (ML) rule to decide between these two hypotheses, i.e., hypothesis \( H_0 \) or \( H_1 \).

**Hint:** Write out the distributions of \( (Y_1, \ldots, Y_n) \) under the two hypotheses.

Problem 2

[Exponential Detection (10pts + 7pts Bonus)]
Consider the following channel:

\[ Y = X + Z, \]

where \( Z \) is independent of \( X \) and the probability density function of \( Z \) is

\[ p_Z(z) = \frac{1}{2} e^{-|z|}. \]
(a) If $\mathbb{P}[X = -1] = q$, $\mathbb{P}[X = 1] = 1 - q$, find the decision regions of the MAP decoder for this channel.

(b) If the prior $q$ is unknown, find the minmax detection rule $\min_{\mathcal{H}} \mathbb{P}_{\mathcal{H}}(q)$ for this channel. Please do not just state the decision rule, but prove the result.

**Hint:** You can use a relation between $\mathbb{P}[\text{error}|X = 1]$ and $\mathbb{P}[\text{error}|X = -1]$.

(c) Now assume that we make two observations

\[
Y_1 = X + Z_1 \\
Y_2 = X + Z_2,
\]

where $Z_1$ and $Z_2$ are independent and identically distributed with densities

\[
p_{Z_1}(z) = p_{Z_2}(z) = \frac{1}{2} e^{-|z|}.
\]

Let $\mathbb{P}[X = -1] = \mathbb{P}[X = 1] = \frac{1}{2}$. Prove that the optimal decision rule for this channel is given by the decision regions in Figure 2.

![Figure 2: Decision regions.](image)

**Problem 3**

[Colored Passband Process (14pts)]

Suppose we have access to a real passband process

\[
x(t) = x_I(t) \cos(\omega_c t) - x_Q(t) \sin(\omega_c t)
\]
where $x_I(t)$ and $x_Q(t)$ are zero-mean jointly Gaussian random processes, with
\[
\mathbb{E}[x_I(t)x_I(t - \tau)] = e^{-2|\tau|} = \mathbb{E}[x_Q(t)x_Q(t - \tau)].
\]

Further, let $x_I(t)$ and $x_Q(t)$ be independent.

\[8pts\] \textbf{(a)} Suppose we want to produce a baseband process $y_{bb}(t)$ such that
\[y_{bb}(t) = y_I(t) + jy_Q(t)\]
with
\[\mathbb{E}[y_I(t)y_I(t - \tau)] = \frac{1}{2}\delta(\tau) = \mathbb{E}[y_Q(t)y_Q(t - \tau)].\]
and $y_I(t)$ and $y_Q(t)$ are zero-mean random processes. Show how to produce such a baseband process using $x(t)$.

\[6pts\] \textbf{(b)} Now if we want to produce a passband process $z(t)$ such that
\[z(t) = z_I(t)\cos(\tilde{\omega}_c t) - z_Q(t)\sin(\tilde{\omega}_c t)\]
with a given $\tilde{\omega}_c > \omega_c$. Furthermore $z_I(t)$ and $z_Q(t)$ to be zero-mean independent Gaussian processes with auto-correlation functions
\[\mathbb{E}[z_I(t)z_I(t - \tau)] = e^{-3|\tau|} = \mathbb{E}[z_Q(t)z_Q(t - \tau)].\]
Starting from $x(t)$, show how we can produce the desired $z(t)$.

\textbf{Hint:} For both parts (a) and (b), you can give a procedure in the frequency domain.

\textbf{Problem 4}

\textbf{[LINEAR PREDICTION (13pts)]}

Given observation
\[y_k = x_k + z_k\]
where $\{z_k\}$ is an iid Gaussian random process with zero-mean and unit variance which is independent of $x_k$. Let $\{x_k\}$ be zero-mean with $\mathbb{E}[x_kx_{k-\ell}] = r_x(\ell)$.

\[9pts\] \textbf{(a)} We want to produce $\hat{x}_k$, given observation $\{y_l\}_{l=-\infty}^{k-1}$ using a linear predictor
\[\hat{x}_k = \sum_{n=1}^{\infty} a_n y_{k-n}.\]
Find an expression for the best MMSE linear predictor for $\hat{x}_k$, i.e., assuming the Paley-Wiener condition holds for all the spectra involved. That is, find $\{a_n\}_{n=1}^{\infty}$ in terms (of a function) of $S_x(D)$.

\[4pts\] \textbf{(b)} If $r_x(\ell) = e^{-2|\ell|}$, where $l$ is an integer, find an explicit expression for the linear predictor.
Problem 5

[RELAY ESTIMATION (15pts)]

Suppose Alex had a wireless transmitter in the INR building and wants to communicate with Carol in the BC building. However, Alex has a friend Bob who is in the PSE building and is willing to help Alex with his mission. Both Carol and Bob have additive Gaussian noise channels.

\[ y_B(t) = x_A(t) + z_B(t) \]
\[ y_C(t) = x_A(t) + x_B(t) + z_C(t) \]

where \( z_B(t) \) and \( z_C(t) \) are independent zero-mean white Gaussian processes with power spectral density \( N_0 \). Now, Alex sends out the sequence \( \{x[n]\} \) using a basis function \( \phi(t) \)

\[ x_A(t) = \sum_n x[n] \phi(t - nT), \]

where \( x[n] \) is an i.i.d. sequence with variance \( \mathcal{E}_x \). Bob after listening to Alex’s message uses a matched filter to collect sufficient statistics and obtains

\[ y_B[k] = x_A[k] + z_B[k]. \]  \hspace{1cm} (1)

He then forwards the symbol \( y[k] \) at the next transmission and therefore sends

\[ x_B(t) = \sum_k y_B[k] \phi(t - (k + 1)T) \]

as the transmitted waveform (see Figure 3).

You can assume the shifted versions of the basis are orthonormal

\[ <\phi_0, \phi_k> = \int \phi^*(t) \phi(t - kT) dt = \delta_k = \int \phi^*(t - (n - k)T) \phi(t - nT) dt. \]

[4pts] (a) Find the spectrum \( S_{z_B}(D) \) of \( z_B[k] \) in (1) after the matched filter operation by Bob.

[7pts] (b) Now Carol receives the superposition of the signal from Alex and Bob, and uses a matched filter receiver to the received signal to obtain a set of sufficient statistics of the input sequence \( x[k] \). Express the output of the matched filter receiver, \( y_C[k] \), in terms of the crosscorrelation of the shifted basis function.

[4pts] (c) Now, Carol wants to use a linear filter \( w[k] \) to estimate \( x[k] \), which minimizes the estimation error

\[ \mathbb{E} \left[ x[k] - w[k] * y_C[k] \right]^2 \]

between the estimate and the signal sent by Alex. Find the optimal filter \( w(k) \).
Problem 6

[ Performance of Equalizers (16pts + 3pts Bonus) ]

Recall the discrete-time channel after matched filtering and sampling:

\[ Y(D) = ||p||Q(D)X(D) + Z(D). \]

Recall also the definition

\[ SNR_{MFB} = \frac{\mathcal{E}_x||p||^2}{N_0}. \]

[5pts] (a) We proved in class that the MMSE-DFE uses the spectral factorization

\[ Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 G(D)G^*(D^{-*}), \tag{2} \]

where \( G(D) \) is causal, strictly stable and monic. Remember that since \( q_k = \langle \tilde{\varphi}_0, \tilde{\varphi}_k \rangle \), we can conclude that \( q_0 = 1 \). Using this, prove that

\[ 1 + \frac{1}{SNR_{MFB}} = \gamma_0 ||g||^2, \]

where

\[ ||g||^2 = \sum_{n=0}^{\infty} |g_n|^2. \]

[3pts] (b) Now, show that for any \( G(D) \) satisfying equation (2), \( ||g||^2 \geq 1 \), with equality if and only if \( Q(D) = 1 \).

Therefore using this we have

\[ \gamma_0 \leq 1 + \frac{1}{SNR_{MFB}}. \]

[5pts] (c) Now, the MMSE-LE is given by (as shown in class)

\[ W_{MMSE-LE}(D) = \frac{1}{||p|| \left( Q(D) + \frac{1}{SNR_{MFB}} \right)}. \]

We also showed in class that for the MMSE-LE, the error spectrum is

\[ S_{EE}^{MMSE-LE}(D) = \frac{N_0}{||p||^2 Q(D) + \frac{1}{SNR_{MFB}}} \]

\[ = \frac{N_0}{||p||^2} W_{MMSE-LE}(D). \]

Hence

\[ \sigma_{MMSE-LE}^2 = \frac{N_0}{||p||^2} w_{MMSE-LE}[0] = \frac{N_0}{||p||^2} D^{-1} \left[ Q(D) + \frac{1}{SNR_{MFB}} \right]_0 \]

(see the instructions page for the notation). Now,

\[ Q(D) + \frac{1}{SNR_{MFB}} = \gamma_0 G(D)G^*(D^{-*}) \]

and

\[ \frac{1}{Q(D) + \frac{1}{SNR_{MFB}}} = \beta L(D)L^*(D^{-*}), \]
where $\beta = \frac{1}{\gamma_0}$ and $L(D) = \frac{1}{\bar{g}(D)}$.

Using this, prove that

$$D^{-1} \left[ \frac{1}{Q(D) + \frac{1}{SNR_{MFB}}} \right] \geq \frac{1}{\gamma_0},$$

(3)

[3pts] (d) We have shown in class that for the MMSE-DFE,

$$S_{EE}^{MMSE-DFE}(D) = \frac{N_0}{\gamma_0||p||^2}.$$  

Hence

$$\sigma_{MMSE-DFE}^2 = \frac{N_0}{\gamma_0||p||^2}.$$  

Using this and inequality (3) show that

$$SNR_{MMSE-LE,U} \leq SNR_{MMSE-DFE,U},$$

with equality iff $Q(D) = 1$.

[Bonus: 3pts] (e) Use results from (b) and (c) to show that

$$SNR_{MMSE-LE,U} \leq SNR_{MMSE-DFE,U} \leq SNR_{MFB}.$$