Orthogonal Filter Banks

Ali Hormati, Amina Chebira and Martin Vetterli

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Outline

- Introduction
- Lowpass channel and projection
- Highpass channel
- The perfect reconstruction filter bank
- Properties
- Design

Goal:
- See the basic building block of time-frequency analysis, the two channel filter bank

Readings:
- Chapter 7, of Fourier and Wavelet Signal Processing
The Haar Case

- Basis vectors are filter impulse responses
  \[ g_n = \varphi_{0,n} = \frac{1}{\sqrt{2}}(\delta_n + \delta_{n-1}) \]
  \[ h_n = \varphi_{1,n} = \frac{1}{\sqrt{2}}(\delta_n - \delta_{n-1}) \]

- Output as linear combination of basis vectors
  \[ x_n = \sum_{k \in \mathbb{Z}} \alpha_k \varphi_{2k,n} + \sum_{k \in \mathbb{Z}} \beta_k \varphi_{2k+1,n} \]
  \[ = \sum_{k \in \mathbb{Z}} \alpha_k g_{n-2k} + \sum_{k \in \mathbb{Z}} \beta_k h_{n-2k} \]

- where
  \[ \langle x, \varphi_{2k} \rangle = \langle x_n, g_{n-2k} \rangle_n = \alpha_k \]
  \[ \langle x, \varphi_{2k+1} \rangle = \langle x_n, h_{n-2k} \rangle_n = \beta_k \]
The Haar Case

- Synthesis

\[
\begin{bmatrix}
\vdots \\
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\vdots \\
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & \alpha_0 \\
1 & -1 & 0 & 0 & 0 & 0 & \cdots & \beta_0 \\
0 & 0 & 1 & 1 & 0 & 0 & \cdots & \alpha_1 \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots & \beta_1 \\
0 & 0 & 0 & 0 & 1 & 1 & \cdots & \alpha_2 \\
0 & 0 & 0 & 0 & 1 & -1 & \cdots & \beta_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix} \Phi^T
\]

- Or

\[x = \Phi \Phi^T x \implies \Phi \Phi^T = I\]

- Note:

\[V = \text{span}(\{\varphi_{0,n-2k}\}_{k \in \mathbb{Z}}) = \text{span}(\{g_{n-2k}\}_{k \in \mathbb{Z}})\]
\[W = \text{span}(\{\varphi_{1,n-2k}\}_{k \in \mathbb{Z}}) = \text{span}(\{h_{n-2k}\}_{k \in \mathbb{Z}})\]

General Orthogonal Filter Bank

- Synthesis

\[
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\vdots \\
\end{bmatrix} = \begin{bmatrix}
\alpha_0 \\
\beta_0 \\
\alpha_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2 \\
\vdots \\
\end{bmatrix} \begin{bmatrix}
\frac{g_n}{\langle x_n, h_n \rangle} \\
\frac{g_{n-2}}{\langle x_n, h_{n-2} \rangle} \\
\frac{g_{n-4}}{\langle x_n, h_{n-4} \rangle} \\
\vdots \\
\end{bmatrix} = \begin{bmatrix}
\alpha_0 \\
\beta_0 \\
\alpha_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2 \\
\vdots \\
\end{bmatrix} \Phi^T
\]

- Analysis:

\[
\begin{bmatrix}
\langle g_n, g_{n-2k} \rangle_n = \delta_k, & \langle h_n, h_{n-2k} \rangle_n = \delta_k, & \langle g_n, h_{n-2k} \rangle_n = 0
\end{bmatrix}
\]

4/7/2011  Orthogonal Two-Channel Filter Banks
General Orthogonal Filter Bank

- Synthesis: Spans

\[ V = \text{span}\{\varphi_{0,-2k}\}_{k \in \mathbb{Z}} = \text{span}\{g_{-2k}\}_{k \in \mathbb{Z}} \]
\[ W = \text{span}\{\varphi_{1,-2k}\}_{k \in \mathbb{Z}} = \text{span}\{h_{-2k}\}_{k \in \mathbb{Z}} \]

- Projections:

\[ x_V = \sum_{k \in \mathbb{Z}} \alpha_k g_{-2k} \]
\[ x_W = \sum_{k \in \mathbb{Z}} \beta_k h_{-2k} \]

- Perfect reconstruction \( \ell^2(\mathbb{Z}) = V \oplus W \)

General Orthogonal Filter Bank: Decomposition
Two Channel – Ideal Filters

Theory of Two-Channel Orthogonal Filter Banks

- The lowpass channel
  \[ l^2(\mathbb{Z}) \]
  \[ x \xrightarrow{\alpha} g_n \xrightarrow{2} \alpha \xrightarrow{2} g_n \rightarrow x_V \]

- Reconstruction
  \[ x_V = \sum_{k \in \mathbb{Z}} \alpha_k g_{n-2k} \]

- Orthogonality of LP and its even shifts
  \[ \langle g_n, g_{n-2k} \rangle = \delta_k \]
  \[ D G^T G U_2 = I \]
  \[ G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2 \]
  \[ |G(e^{j\omega})|^2 + |G(e^{j(\omega+\pi)})|^2 = 2 \]
Theory of Two-Channel Orthogonal Filter Banks

- Autocorrelation of LP filter

\[ D_2 A U_2 = I \]

\[ A(z) + A(-z) = 2 \]

\[ A(e^{j\omega}) + A(e^{j(\omega + \pi)}) = 2 \]

\[ A(z) = 1 + 2 \sum_{k=0}^{\infty} \alpha_{2k+1}(z^{2k+1} + z^{-(2k+1)}) \]

- This leads to a design procedure
  - Find such an autocorrelation
  - Use spectral factorization

Theory of Two-Channel Orthogonal Filter Banks

- Orthogonal Projection Property of the Lowpass Channel

- Composition of four linear operators (infinite matrices):

\[ x_V = P_V x = G U_2 D_2 G^T x \]

- check idempotency and self-adjointness,

\[ P_V^2 = (G U_2 D_2 G^T)(G U_2 D_2 G^T) = G U_2 D_2 G^T = P_V \]

\[ P_V^T = (G U_2 D_2 G^T)^T = G (U_2 D_2)^T G^T = G U_2 D_2 G^T = P_V \]

- indeed, is an orthogonal projection operator, with the range:

\[ V = \text{span}\{g_{n-2k}\}_{k \in \mathbb{Z}} \]

(6.19)
Theory of Two-Channel Orthogonal Filter Banks

- The highpass channel

\[
\begin{array}{c}
\begin{array}{c}
\downarrow 2 \\
\beta \\
\downarrow 2 \\
\end{array}
\end{array}
\begin{array}{c}
h_{-n} \\
h_n \\
x_W
\end{array}
\]

- Reconstruction

\[x_W = \sum_{k \in \mathbb{Z}} \beta_k h_{n-2k}\]

- Orthogonality of HP and its even shifts

\[
\begin{align*}
\langle h_n, h_{n-2k} \rangle &= \delta_k \\
H(z)H(z^{-1}) + H(-z)H(-z^{-1}) &= 2 \\
\left|H(e^{j\omega})\right|^2 + \left|H(e^{j(\omega+\pi)})\right|^2 &= 2
\end{align*}
\]

Theory of Two-Channel Orthogonal Filter Banks

- Orthogonality of LP and HP filters

\[
\begin{align*}
\langle g_n, h_{n-2k} \rangle &= 0 \\
\langle g_n, h_{n-2k} \rangle &= c_{2k} = 0
\end{align*}
\]

- Cross-correlation view:

\[
\begin{align*}
\langle g_n, h_{n-2k} \rangle &= 0 \\
\langle g_n, h_{n-2k} \rangle &= c_{2k} = 0
\end{align*}
\]
Theory of Two-Channel Orthogonal Filter Banks

• Orthogonal filter banks implement an ONB

  • Assume \( \|g\| = 1, \langle g_n, g_{n-2k} \rangle = \delta_k \) and moreover, \( g_n \) is of even length \( L \). By choosing

\[
    h_n = (-1)^n g_{L-1-n} \quad \xrightarrow{\mathcal{ZT}} \quad H(z) = -z^{-L+1} G(-z^{-1})
\]

Then, \( \{g_{n-2k}, h_{n-2k}\}_{k \in \mathbb{Z}} \) makes an ONB for \( \ell_2(\mathbb{Z}) \).

• Proof: See the board!

Polyphase View of Orthogonal FBs (on board)
Polynomial Approximation by FB

- Assume \( h_n = \delta_n - \delta_{n-1} \) or equivalently, \( H(z) = 1 - z^{-1} \)
- It reduces the degree of polynomials by 1 (proof on board)

\[
x_n = \sum_{k=0}^{N} a_k n^k \quad x_n * h_n = \sum_{k=0}^{N-1} b_k n^k
\]

- Having \( N \) zeros at 1, reduces the degree by \( N - 1 \)

\[
G(z) = (1 + z^{-1})^N R(z) \quad \rightarrow \quad H(z^{-1}) = z^{L-1}(1 - z^{-1})^N R(-z)
\]

- So, OFB with the above \( G(z) \) kills polynomials up to degree \( N - 1 \) in the highpass branch (Caution: boundary effects).

Design of Orthogonal Two-Channel Filter Banks

- **Design Facts:**
  - Just design \( G(z) \), the rest follows!
  - Start from a deterministic autocorrelation \( A(z) \)
    \[
    A(z) + A(-z) = 2
    \]
  - Do spectral factorization
    \[
    A(z) = G(z)G(z^{-1})
    \]

- **Methods:**
  - Polynomial Approximation Design
  - Lowpass Approximation Design
  - Lattice Factorization Design
Wrap Up

Filter banks: Swiss army knife of TF analysis!

Theory
• Synthesis LP orthogonal to its even shifts
• All else follows!
• HP: modulated, time reversed LP filter
• Analysis: time reversed synthesis filters
• Orthonormal basis for $l_2(z)$.
• Polynomial approximation when LP has roots at $z=-1$

Design
• Windowing
• ACF factorization
• Polynomial approximation