A Windowed Graph Fourier Transform

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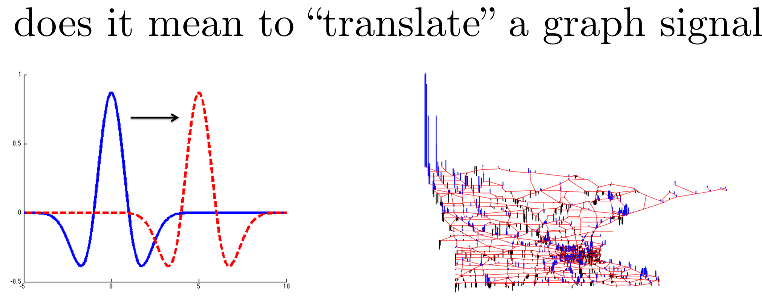
Motivation

- High-dimensional data naturally reside on graphs in social, electricity, transportation, and sensor networks
- Weighted graphs are also a flexible tool to describe topologically-complicated data domains
  - Similarities between data points in statistical learning
  - Functional connectivities between regions of the brain
- A number of multiscale wavelet transforms for signals on graphs have been introduced recently (e.g., [1]-[3])
- **Our objective**: Define a vertex-frequency transform to extract information from high-dimensional signals on weighted graphs (statistically or visually), as well as to regularize ill-posed inverse problems
- Classical time-frequency transforms provide joint descriptions of signals' temporal and spectral behavior
  - Particularly useful for extracting information from signals with localized oscillations
  - E.g., audio processing, vibration analysis, radar detection
- **Essence of the problem**: Weighted graphs are irregular structures that lack a shift-invariant notion of translation
  - What does it mean to "translate" a graph signal?



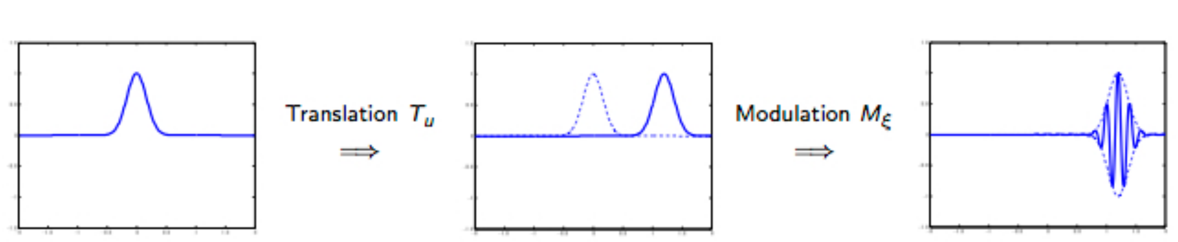
- **Our approach**: Develop generalized notions of convolution, translation, and modulation in the graph setting, and then mimic the classical windowed Fourier transform construction

The Classical Windowed Fourier Transform

- Windowed Fourier atoms:

$$g_{u,\xi}(t) := (M_\xi T_u g)(t) = g(t-u)e^{2\pi i \xi t}$$

- The atoms $g_{u,\xi}$ are localized in time and frequency:

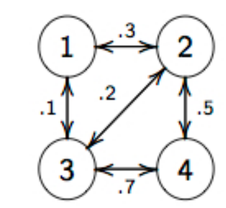


- Windowed Fourier transform of $f \in L^2(\mathbb{R})$:

$$Sf(u,\xi) := \langle f, g_{u,\xi}\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t-u)}e^{-2\pi i \xi t}dt$$

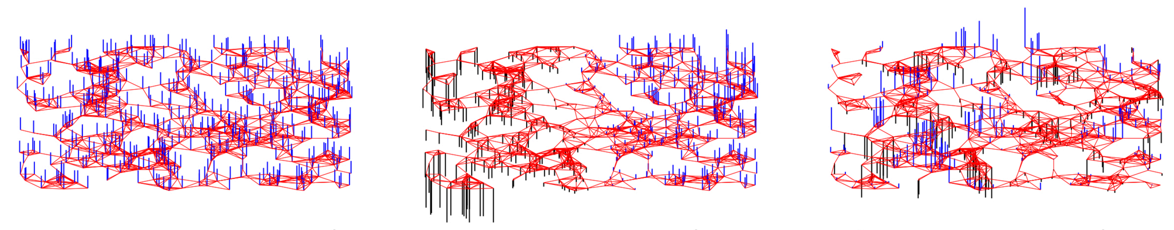
Spectral Graph Theory

- Connected, undirected, weighted graphs: $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$



$$W = \begin{bmatrix} 0 & .3 & .1 & 0 \\ .3 & 0 & .2 & .5 \\ .1 & .2 & 0 & .7 \\ 0 & .5 & .7 & 0 \end{bmatrix} \quad D = \begin{bmatrix} .4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix}$$

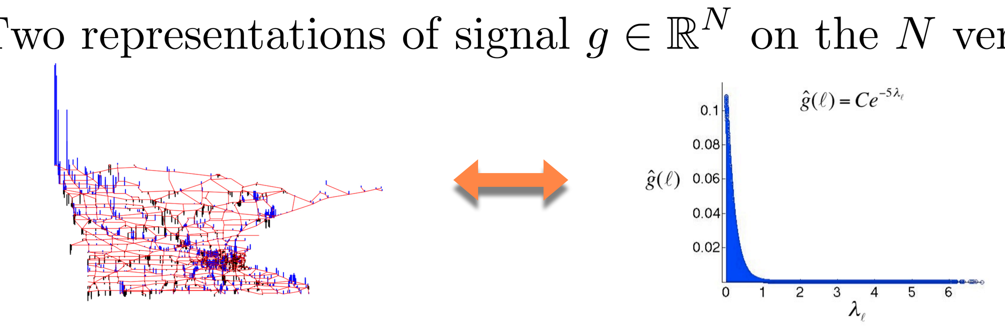
- Non-normalized graph Laplacian: $\mathcal{L} = D - W$
- Complete set of orthonormal eigenvectors and associated real, non-negative eigenvalues: $\mathcal{L}\chi_\ell = \lambda_\ell \chi_\ell$
- Values of eigenvectors associated with lower frequencies vary less rapidly across connected vertices:



- Fourier transform: expansion of a signal in terms of the eigenfunctions of the Laplacian / graph Laplacian

Analog Signals	Graph Signals
$\hat{g}(\xi) = \langle g, e^{2\pi i \xi t}\rangle = \int_{\mathbb{R}} g(t)e^{-2\pi i \xi t}dt$	$\hat{g}(\ell) = \langle g, \chi_\ell \rangle = \sum_{n=1}^{N} g(n)\chi_\ell^*(n)$

- Two representations of signal $g \in \mathbb{R}^N$ on the N vertices:

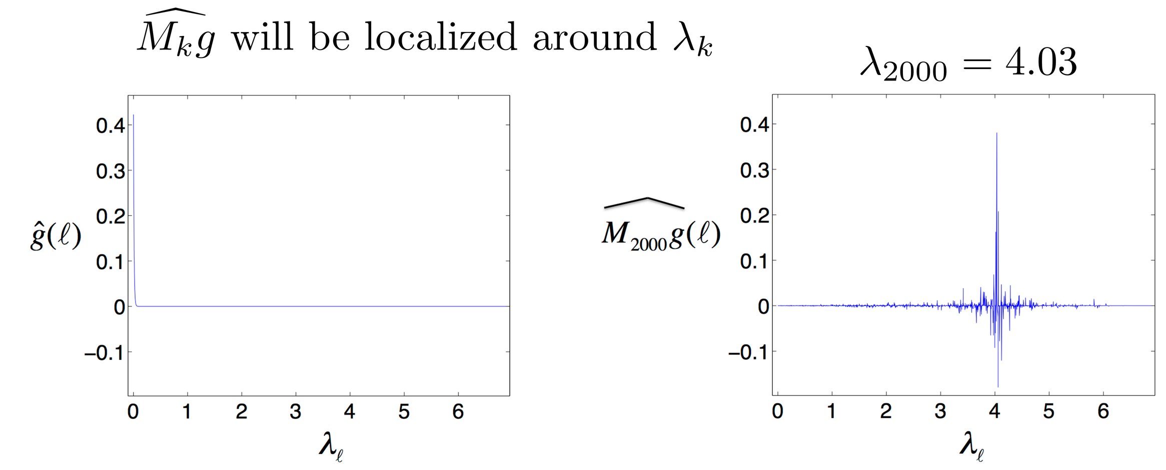


Generalized Modulation

- Define generalized modulation via multiplication by a graph Laplacian eigenvector

Analog Signals	Graph Signals
For $g \in L^2(\mathbb{R})$, $(M_\xi g)(t) := e^{2\pi i \xi t}g(t)$	For $g \in \mathbb{R}^N$, we define $(M_k g)(n) := \sqrt{N}\chi_k(n)g(n)$

- The classical modulation operator represents a translation in the Fourier domain
- Generalized modulation as a graph spectral shift?
  - $\widehat{M_k\delta_0}(\lambda_\ell) = \delta_0(\lambda_\ell - \lambda_k)$
  - Moreover, if $\hat{g}$ is sufficiently localized around 0, then $\widehat{M_k g}$ will be localized around λ_k



Generalized Translation

- Convolution in the time (vertex) domain is multiplication in the Fourier (graph spectral) domain:

Analog Signals

For $f, g, \in L^2(\mathbb{R})$,

$$(f*g)(t) := \int_{\mathbb{R}} f(\tau)g(t-\tau)d\tau,$$

which implies

$$(f*g)(t) = \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\xi)e^{2\pi i \xi t}d\xi$$

Graph Signals

For $f, g \in \mathbb{R}^N$, we define

$$(f*g)(n) := \sum_{\ell=0}^{N-1} \hat{f}(\ell)\hat{g}(\ell)\chi_\ell(n)$$

- Define generalized translation via generalized convolution with a delta:

Analog Signals

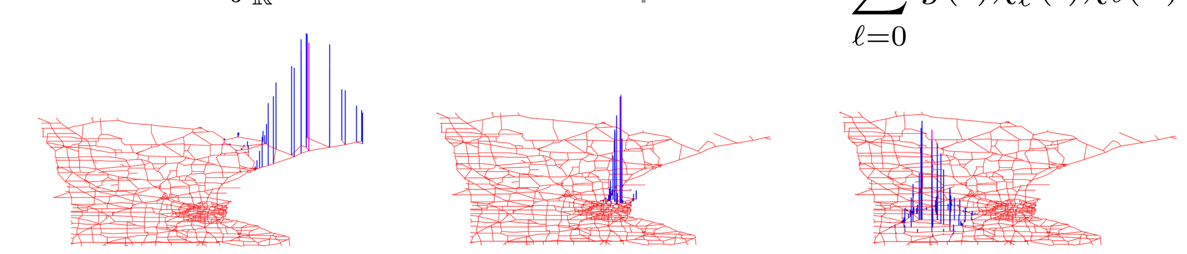
For $g \in L^2(\mathbb{R})$,

$$(T_u g)(t) := g(t-u) = (g*\delta_u)(t) = \int_{\mathbb{R}} \hat{g}(\xi)e^{-2\pi i \xi u}e^{2\pi i \xi t}d\xi$$

Graph Signals

For $g \in \mathbb{R}^N$, we define

$$(T_i g)(n) := \sqrt{N}(g*\delta_i)(n) = \sqrt{N}\sum_{\ell=0}^{N-1}\hat{g}(\ell)\chi_\ell^*(i)\chi_\ell(n)$$



$T_{200}g$ $T_{1000}g$ $T_{2000}g$

- Smoothness of g controls localization of $T_i g$
- Warnings:
  - No group structure
  - Generalized translation operators are not isometric

Windowed Graph Fourier Frames

- Windowed graph Fourier atoms:

$$g_{i,k} := M_k T_i g$$

- Windowed graph Fourier transform:

$$Sf(i,k) := \langle f, g_{i,k}\rangle$$

- If $\hat{g}(0) \neq 0$, then $\{g_{i,k}\}_{i=1,2,\ldots,N;\ k=0,1,\ldots,N-1}$ is a frame:

$$A\|f\|_2^2 \leq \sum_{i=1}^{N}\sum_{k=0}^{N-1}|\langle f, g_{i,k}\rangle|^2 \leq B\|f\|_2^2, \text{ where}$$

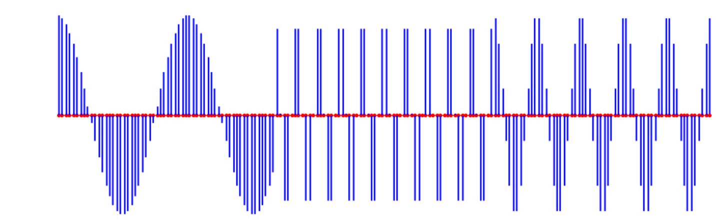
$$A := \min_{i\in 1,2,\ldots,N}\{\|T_i g\|_2^2\} \geq N|\hat{g}(0)|^2 > 0,$$

$$B := \max_{i\in 1,2,\ldots,N}\{\|T_i g\|_2^2\} \leq N^2\mu^2\|g\|_2^2, \text{ and}$$

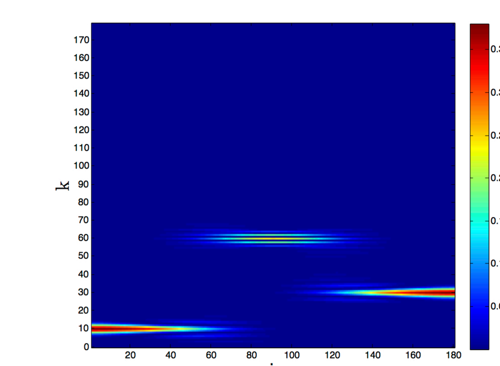
$$\mu := \max_{\substack{\ell\in 0,1,\ldots,N-1 \\ i\in 1,2,\ldots,N}} |\langle \chi_\ell, \delta_i\rangle|.$$

Examples and Discussion

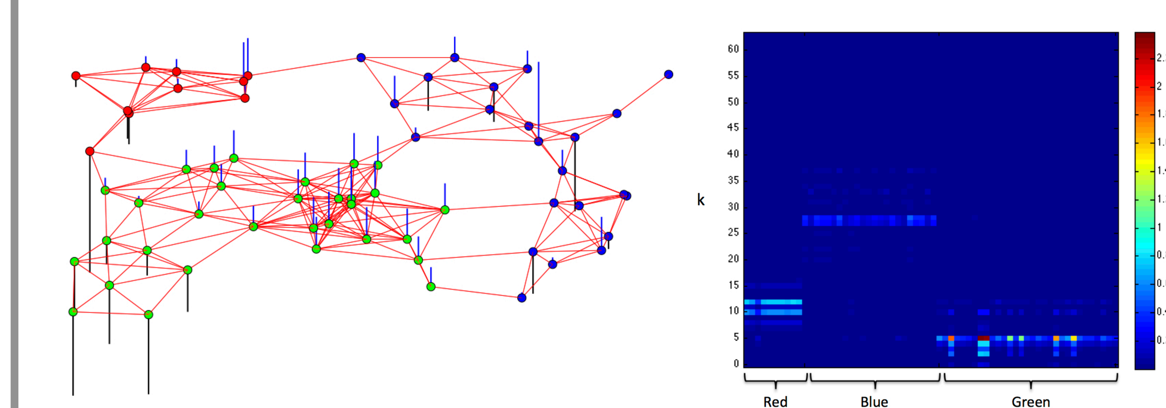
- Signal f on the 180 vertex path graph, composed by summing χ_{10} restricted to the first 60 vertices, χ_{60} restricted to next 60, and χ_{30} restricted to final 60 :



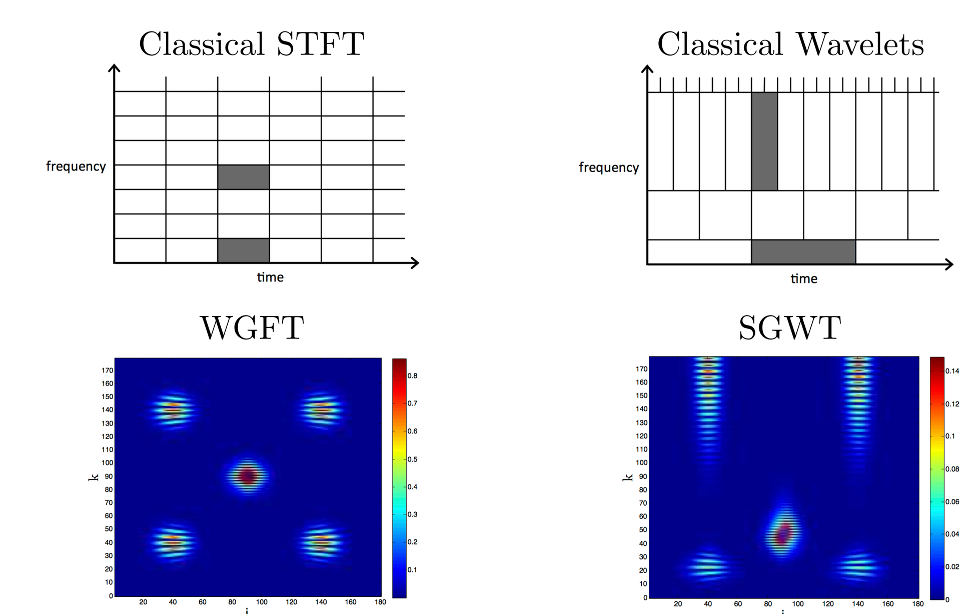
- "Spectrogram" of f showing $|Sf(i,k)|^2$, using a normalized heat kernel window with $\tau = 300$:



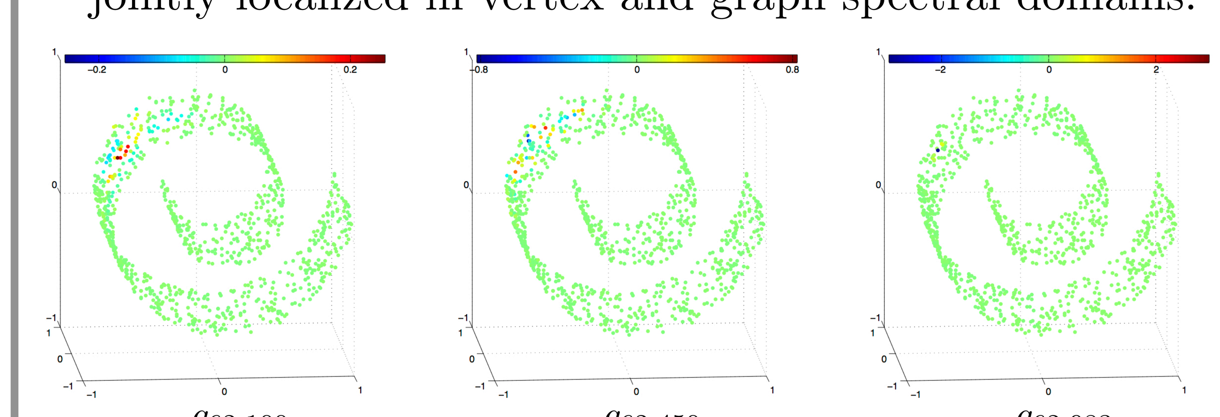
- Signal f comprised of three graph Laplacian eigenvectors $(\chi_{10}, \chi_{27}, \chi_5)$ restricted to three different clusters of a random sensor network:

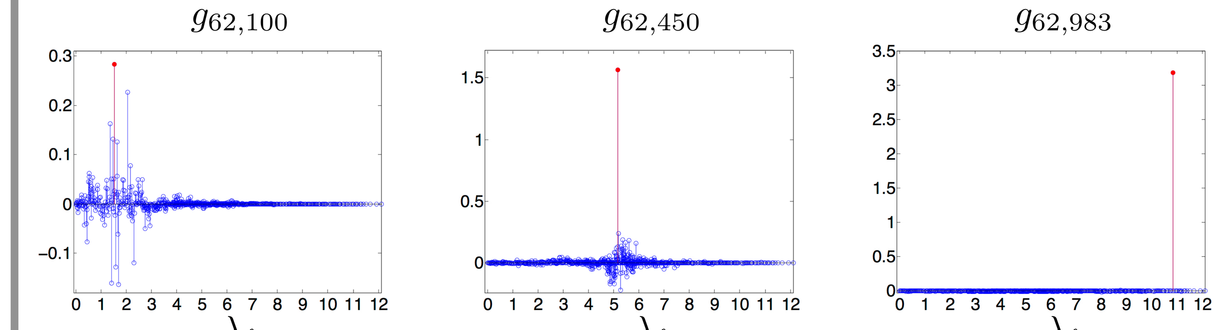


- Tiling comparison on the path graph:



- However, due to localized eigenvectors, atoms may be jointly localized in vertex and graph spectral domains:





[1] R. R. Coifman and M. Maggioni, "Diffusion wavelets," Appl. Comput. Harm. Anal., vol. 21, pp. 53-94, Jul. 2006.
[2] D. K. Hammond, P. Vandergheynst, and R. Gribonval, "Wavelets on graphs via spectral graph theory," Appl. Comput. Harm. Anal., vol. 30, pp. 129-150, Mar. 2011.
[3] S. K. Narang and A. Ortega, "Perfect reconstruction two-channel wavelet filter-banks for graph structured data," IEEE Trans. Signal Process., vol. 60, pp. 2786-2799, Jun. 2012.

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