On the Sparsity of Wavelet Coefficients for Signals on Graphs

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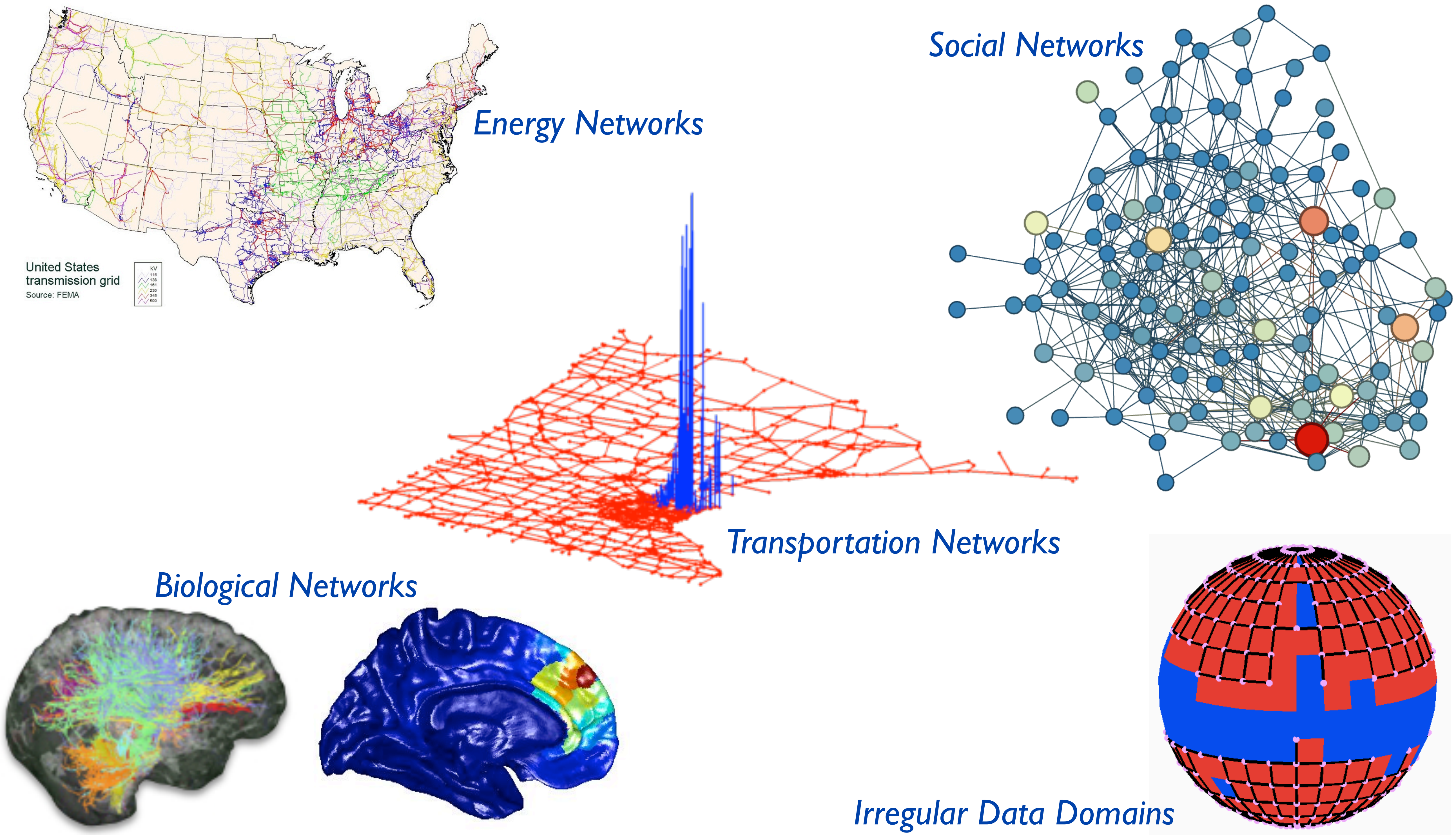
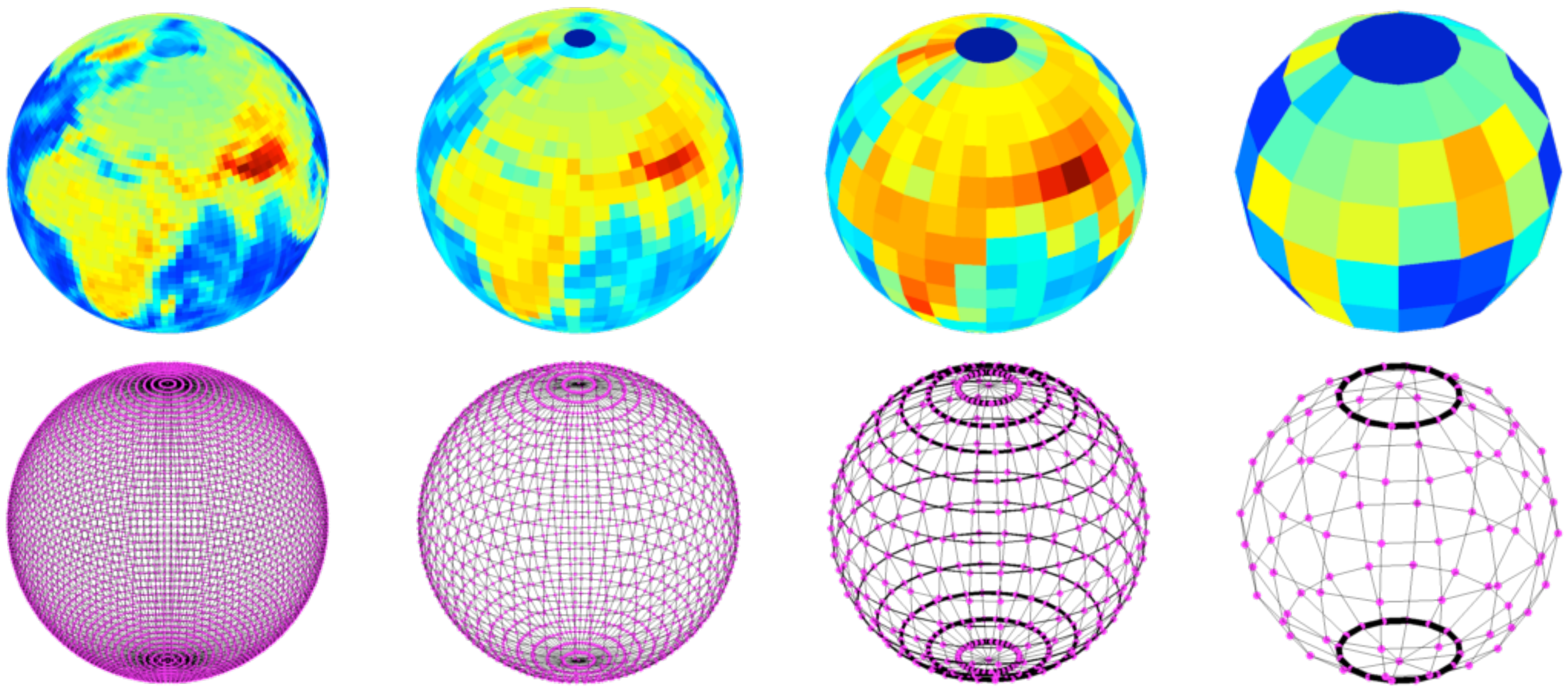
EPFL – Signal Processing Laboratory (LTS2)
http://lts2.epfl.ch

Signal Processing on Graphs

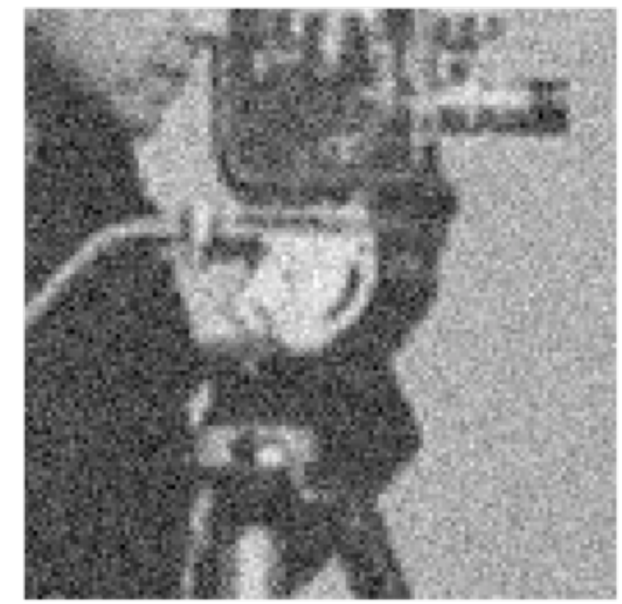


Some Typical Processing Problems

Compression / Visualization

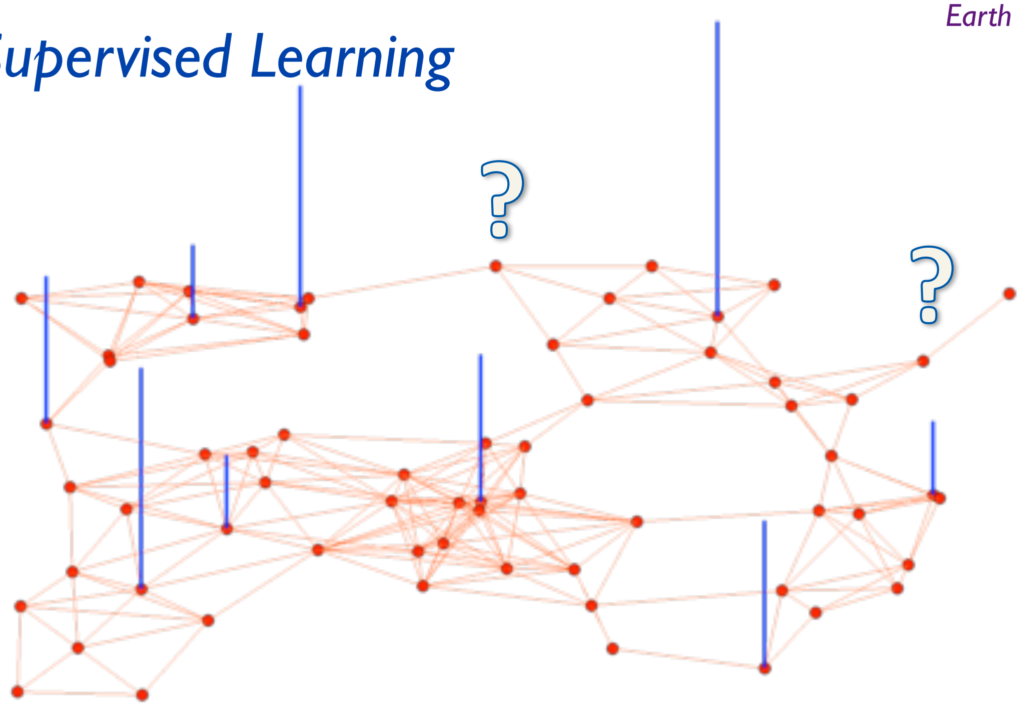


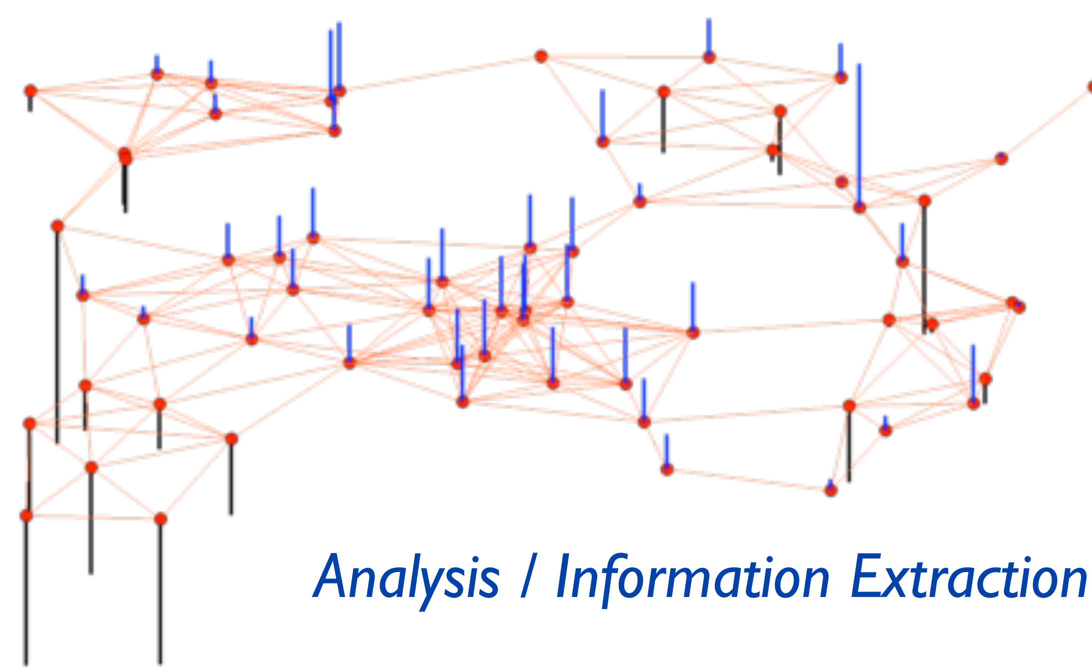
Earth data source: Frederik Simons



Denoising

Semi-Supervised Learning

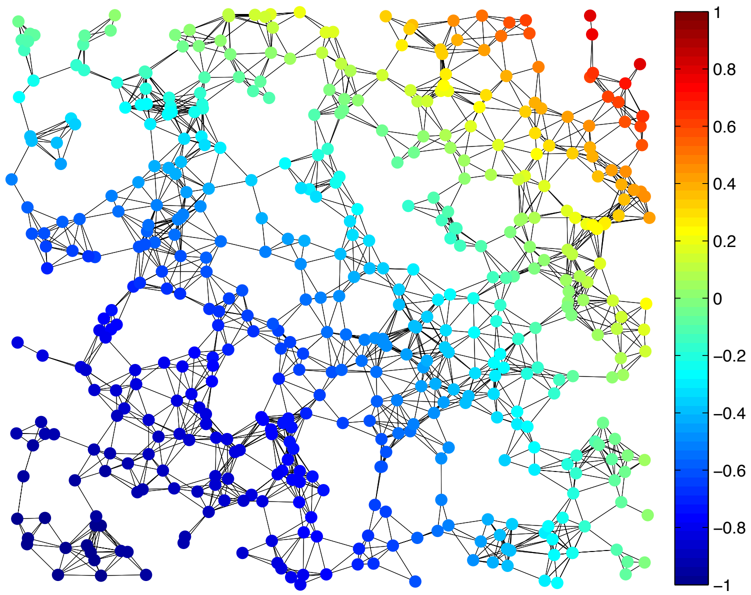




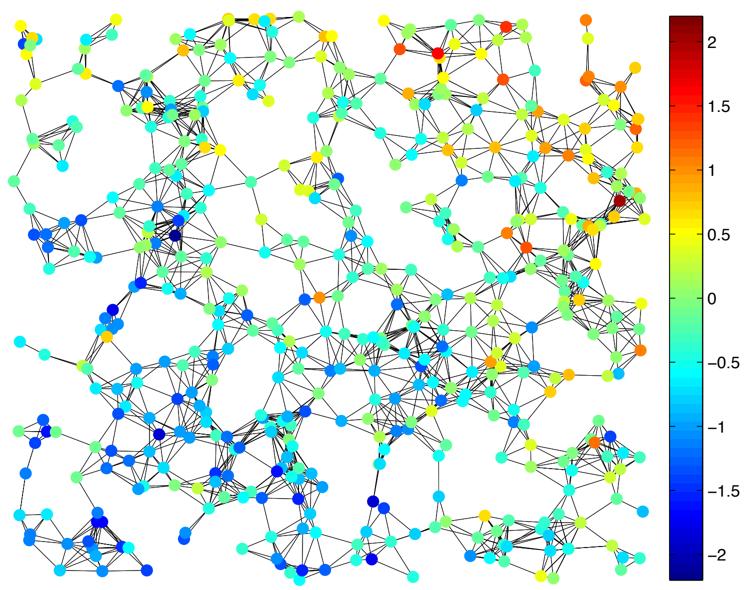
Analysis / Information Extraction

Simple Motivating Examples

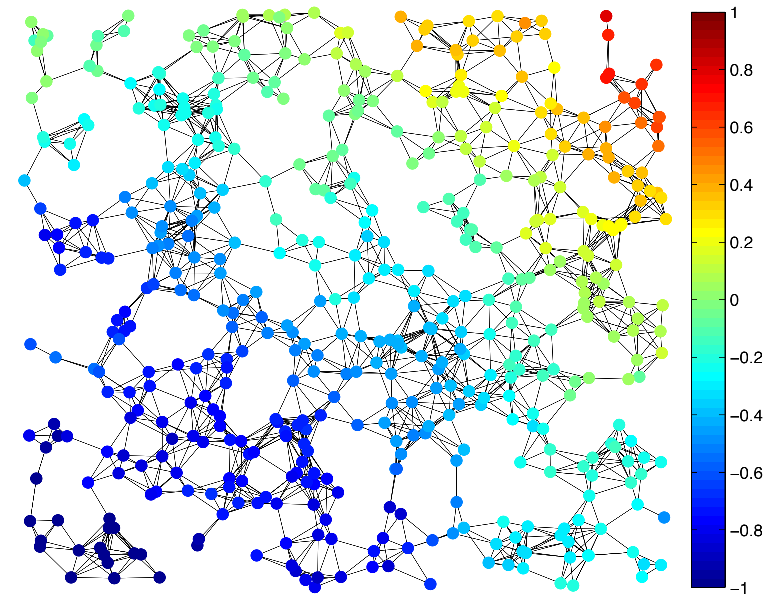
- Tikhonov regularization for denoising: $\operatorname{argmin}_f \left\{ ||f - y||_2^2 + \gamma f^T \mathcal{L} f \right\}$



Original



Noisy



Denoised

Simple Motivating Examples

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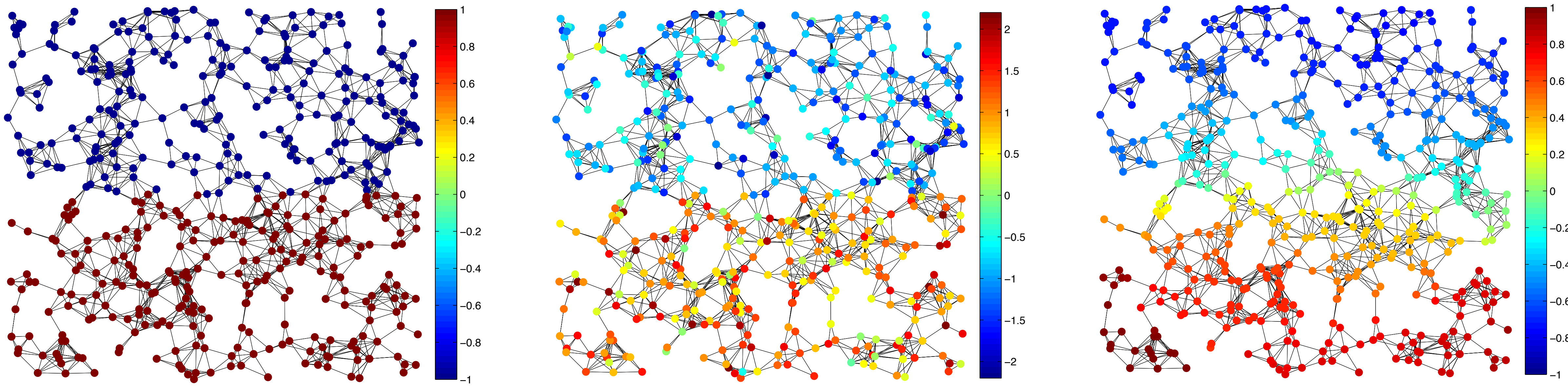
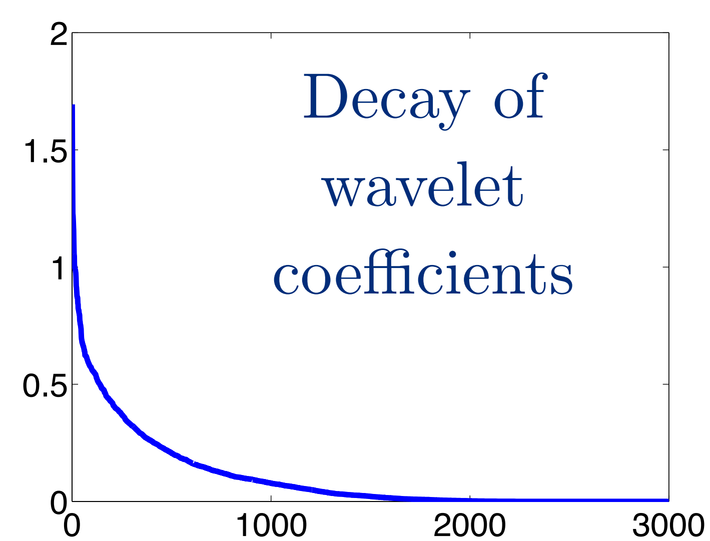
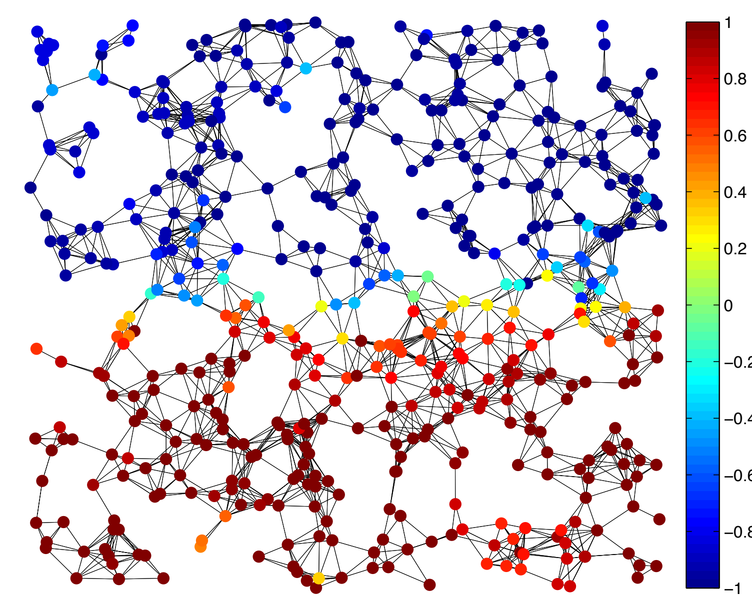


Original | Noisy | Denoised

- Wavelet denoising: $\text{argmin}_a \left\{ ||f - W^* a||_2^2 + \gamma ||a||_{1,\mu} \right\}$





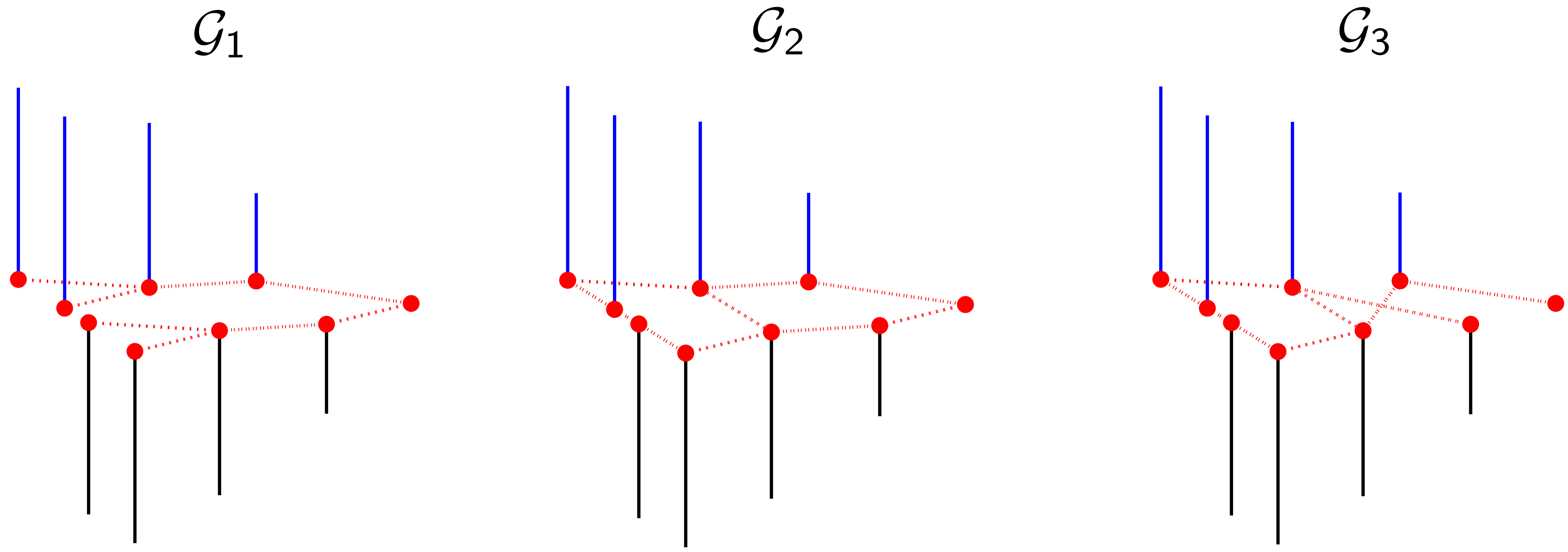
Research Questions

- For signals on Euclidean data domains, we have results characterizing classes of signals that are well-approximated by different transforms
  - e.g., piecewise-smooth 1D signals by wavelets, 2D cartoons with curvilinear discontinuities by curvelets/shearlets

- Which multiscale transforms for signals on graphs are well-suited for which signal processing tasks, which classes of signals, and which types of graphs?

- Connections between properties of graph signals and the decay of their wavelet coefficients?

Smoothness of Graph Signals



To identify and exploit structure in the data, we need to account for the intrinsic geometric structure of the underlying graph data domain

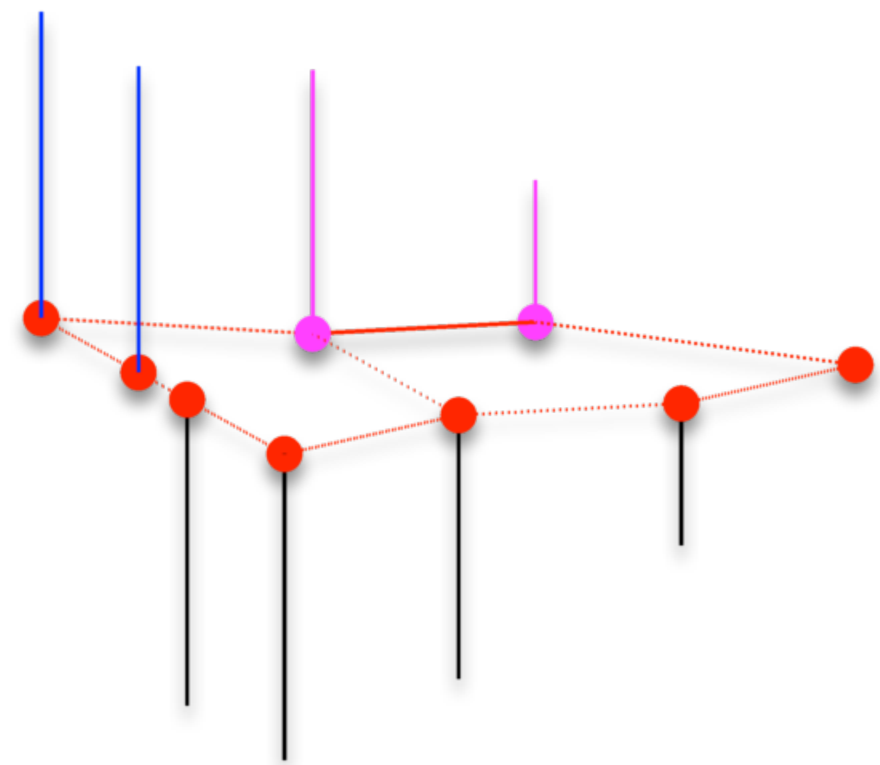
Global Regularity of Graph Signals

Notions of Global Regularity for Graph Signals

Discrete Calculus, Grady and Polimeni, 2010

Edge Derivative

$$\left.\frac{\partial \mathbf{f}}{\partial e}\right|_m := \sqrt{w(m,n)}\,[f(n) - f(m)]$$



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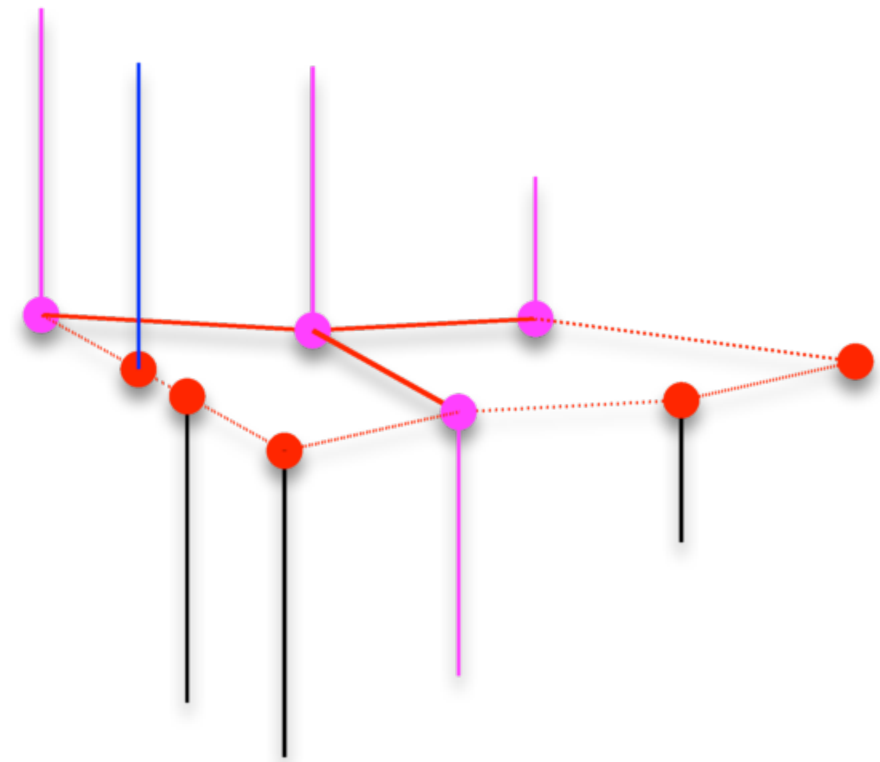

Edge Derivative

$$\left.\frac{\partial \mathbf{f}}{\partial e}\right|_m := \sqrt{w(m,n)}\,[f(n)-f(m)]$$

Graph Gradient

$$\nabla_m \mathbf{f} := \left[\left\{\left.\frac{\partial \mathbf{f}}{\partial e}\right|_m\right\}_{e\in\mathcal{E}\text{ s.t. } e=(m,n)}\right]$$

Local Variation

$$||\nabla_m \mathbf{f}||_2 = \left[\sum_{n\in\mathcal{N}_m} w(m,n)\,[f(n)-f(m)]^2\right]^{\frac{1}{2}}$$

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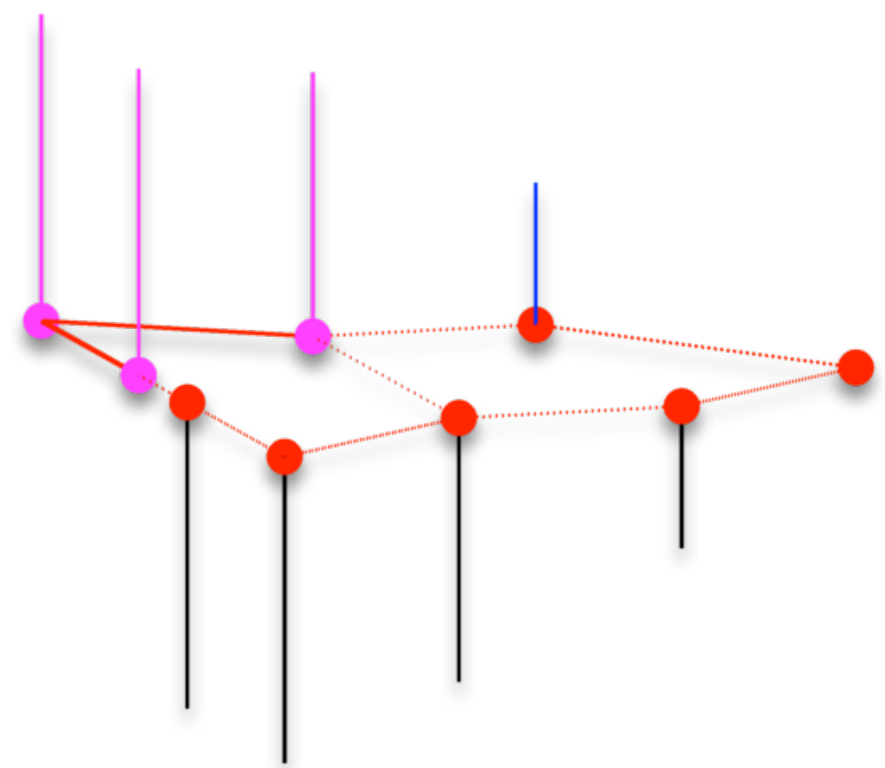

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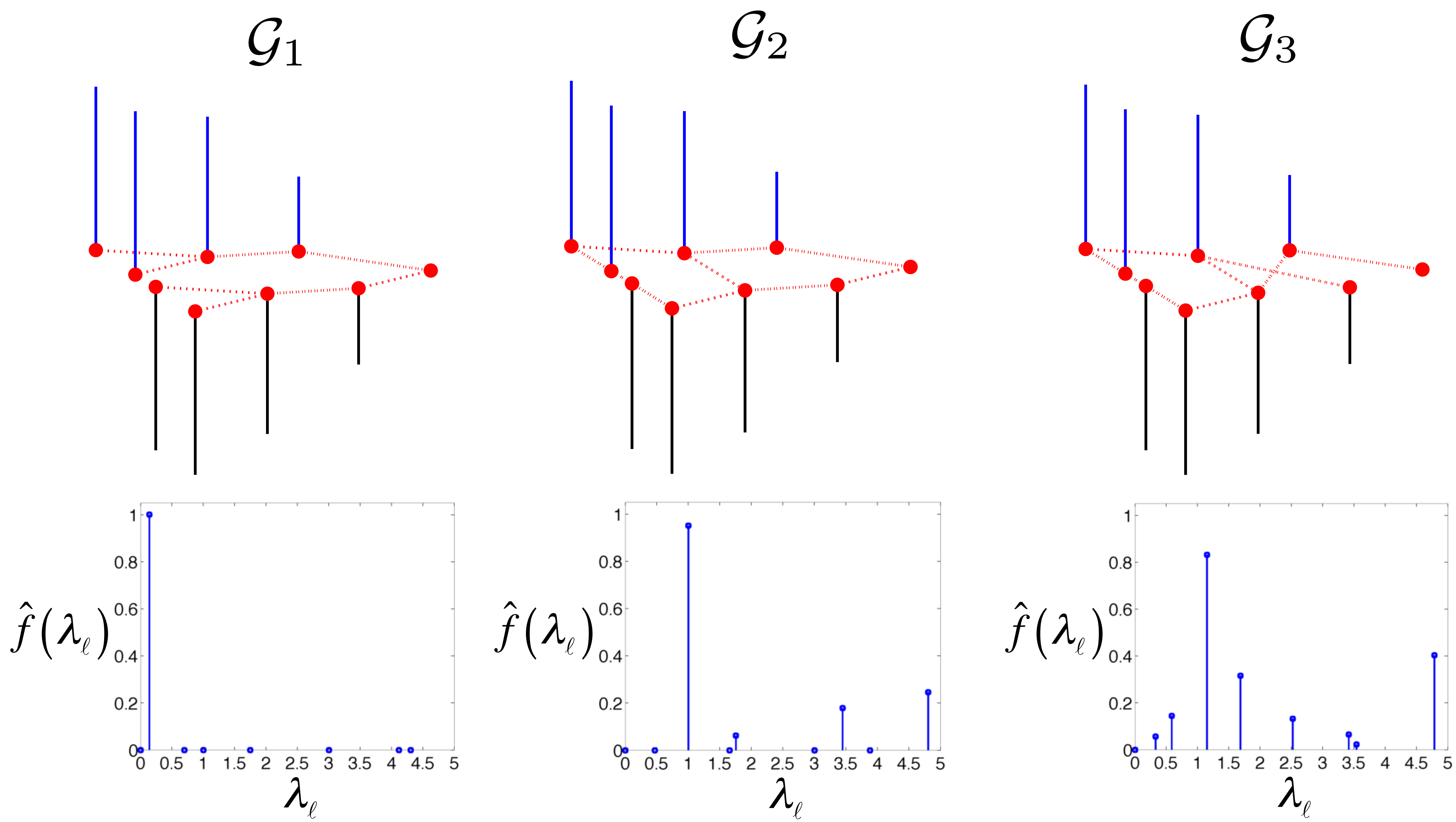
Quadratic Form

$$\frac{1}{2}\sum_{m\in V} ||\nabla_m \mathbf{f}||_2^2 = \sum_{(m,n)\in\mathcal{E}} w(m,n)\,[f(n) - f(m)]^2 = \mathbf{f}^{\mathrm{T}}\mathcal{L}\mathbf{f}$$

$$\Rightarrow \quad ||\mathbf{f}||_{\mathcal{L}} := ||\mathcal{L}^{\frac{1}{2}}\mathbf{f}||_2 = \sqrt{\mathbf{f}^{\mathrm{T}}\mathcal{L}\mathbf{f}}$$

Smoothness of Graph Signals Revisited



$\mathbf{f}^{\mathrm{T}}\mathcal{L}_1\mathbf{f} = 0.14$ $\mathbf{f}^{\mathrm{T}}\mathcal{L}_2\mathbf{f} = 1.31$ $\mathbf{f}^{\mathrm{T}}\mathcal{L}_3\mathbf{f} = 1.81$

Notions of Global Regularity for Graph Signals
Generalizations

p-Dirichlet Form
(Elmoataz et al., 2008)

$$\frac{1}{p}\sum_{m\in V}||\nabla_m \mathbf{f}||_2^p = \frac{1}{p}\sum_{m\in\mathcal{V}}\left[\sum_{n\in\mathcal{N}_m} w(m,n)\left[f(n)-f(m)\right]^2\right]^{\frac{p}{2}}$$

Discrete Sobolev Semi-Norm

$$\|f\|_{\mathcal{H}^p} := \|\mathcal{L}^p f\|_2 = \|\widehat{\mathcal{L}^p f}\|_2 = \sqrt{\sum_{\ell}|\lambda_\ell|^{2p}|\hat{f}(\ell)|^2}$$

- In the continuous setting, the space $\mathbb{W}^p(\mathbb{R})$ of p-times differentiable Sobolev functions are those satisfying

$$\int_{-\infty}^{\infty}|\omega|^{2p}|\hat{f}(\omega)|^2 d\omega < \infty$$

Mallat, 2008, pp. 438-9

- In the graph setting,

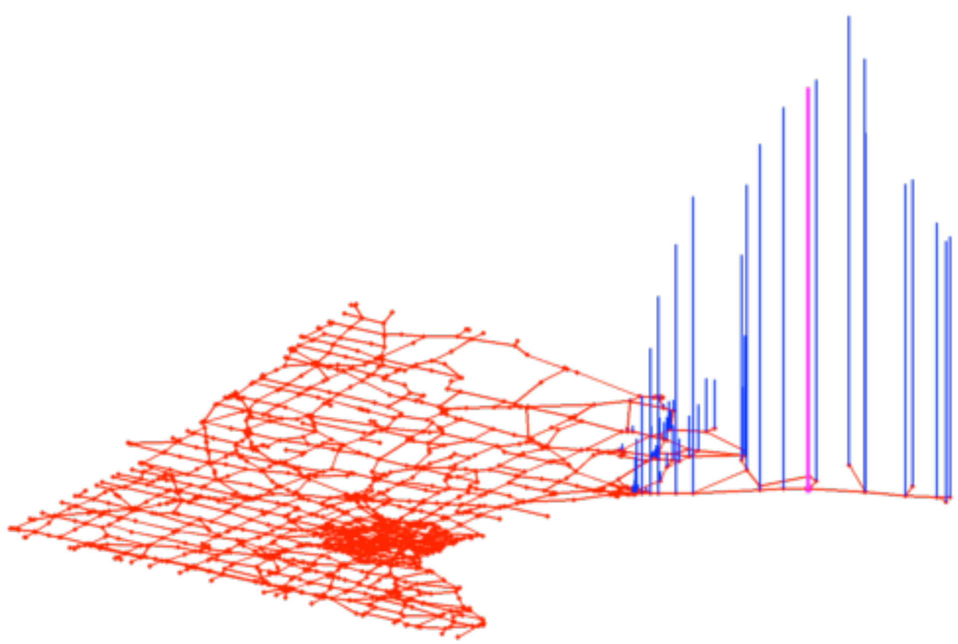
$$\frac{||f||_{\mathcal{H}^p}}{||f||_2} \leq \lambda_{\max}^p \text{ for all } f \in \mathbb{R}^N$$

Wavelet Coefficient Decay of Globally Regular Graph Signals

Spectral Graph Wavelets

Hammond et al., Wavelets on graphs via spectral graph theory, ACHA, 2011

- Generalized translation
  - Classical setting: $(T_s g)(t) = g(t-s) = \int_{\mathbb{R}} \hat{g}(\xi) e^{-2\pi i \xi s} e^{2\pi i \xi t} d\xi$
  - Graph setting: $(T_n g)(i) := \sum_{\ell=0}^{N-1} \hat{g}(\lambda_\ell) u_\ell^*(n) u_\ell(i)$



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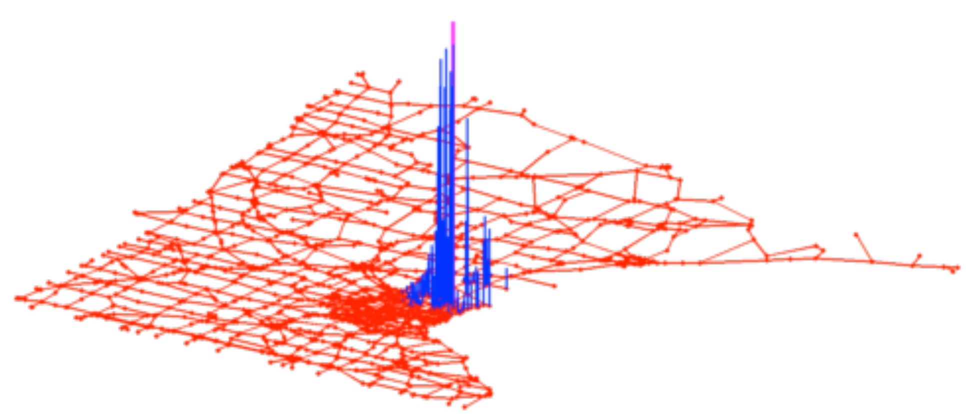

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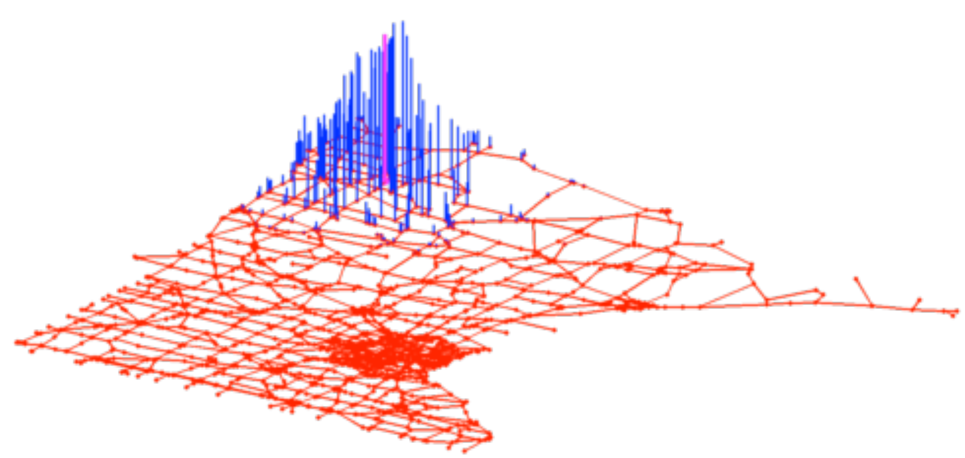

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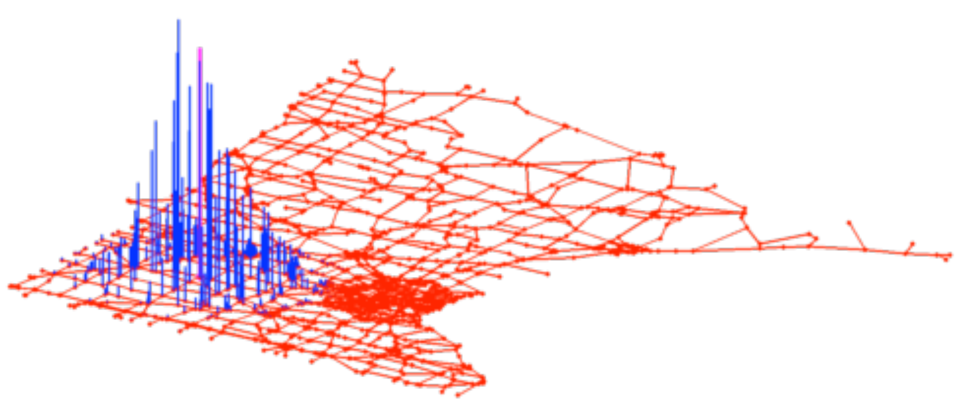

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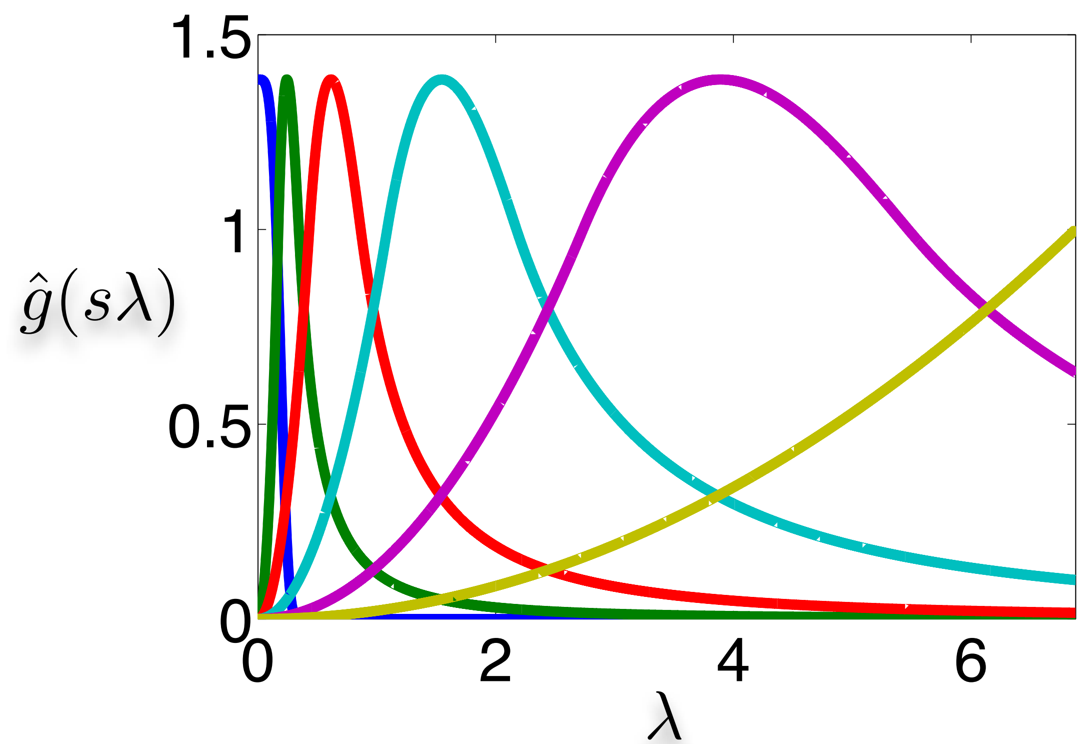
- Generalized dilation: $\widehat{\mathcal{D}_s g}(\lambda) = \hat{g}(s\lambda)$



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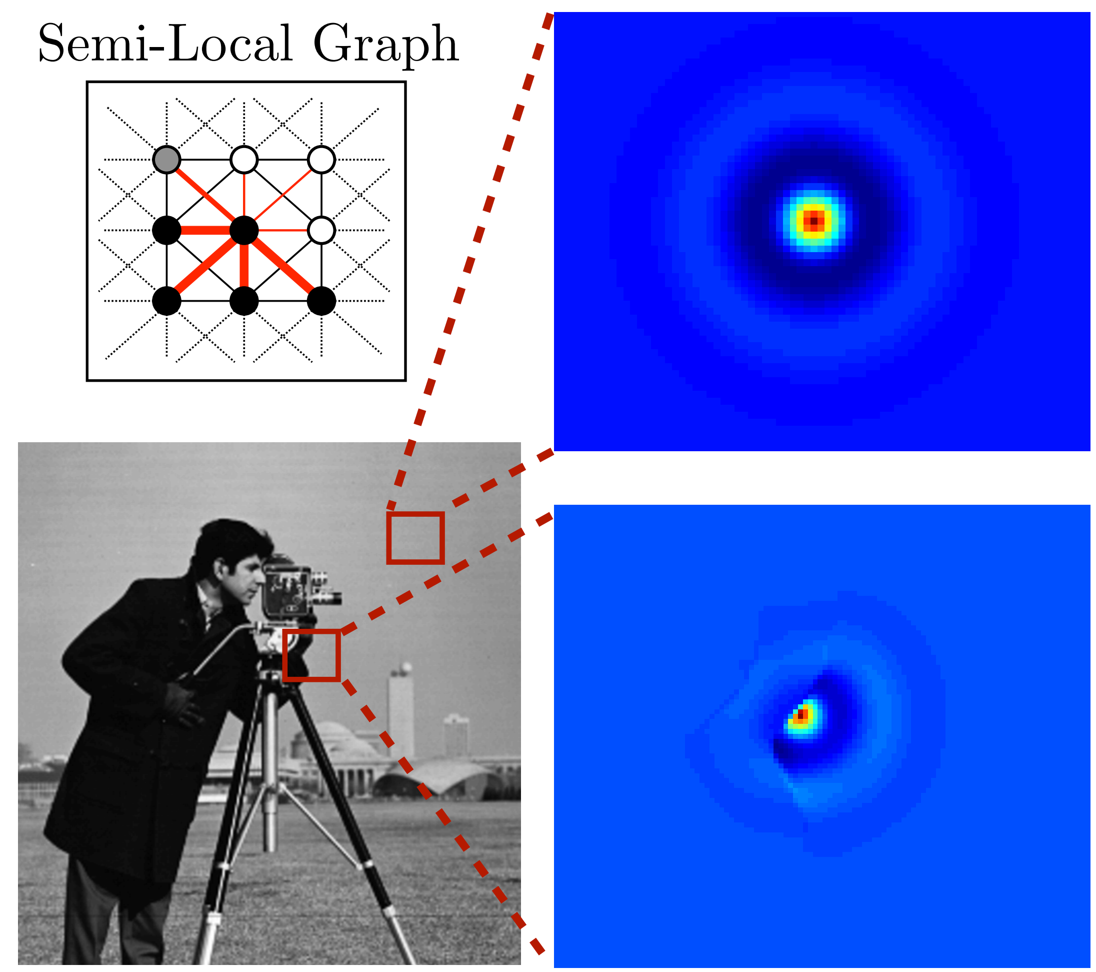


- Generalized dilation: $\widehat{\mathcal{D}_s g}(\lambda) = \hat{g}(s\lambda)$
- Spectral graph wavelet at scale s, centered at vertex n:

$$\psi_{s,n}(i) := (T_n D_s g)(i) = \sum_{\ell=0}^{N=1} \hat{g}(s\lambda_\ell) u_\ell^*(n) u_\ell(i)$$

Wavelet Coefficient Decay of Globally Regular Graph Signals

Proposition 1

Let $p \geq 1$, and assume that $C_p := \int_0^\infty |\hat{g}(s)|^2 / s^{2p} ds < \infty$. Then

$$\int_0^\infty s^{-2p} \sum_n |\langle f, \psi_{s,n} \rangle|^2 ds = C_p ||f||_{\mathcal{H}^{(2p-1)/2}}.$$

Proposition 2

Assume that $\hat{g}(\lambda) = \sum_{k=p}^q a_k \lambda^k$ for some $p \geq 1$ (implying $\hat{g} = 0$) Then

$$|\Psi f(s,n)| = |\langle f, \psi_{s,n} \rangle| \leq \sum_{k=p}^q |a_k| s^k ||f||_{\mathcal{H}^k}.$$

Ongoing Work: Local Regularity and Wavelet Coefficient Decay of Locally Regular Graph Signals

Notions of Local Regularity

Local Variation

$$||\nabla_m \mathbf{f}||_2 = \left[\sum_{n \in \mathcal{N}_m} w(m,n) \left[f(n) - f(m) \right]^2 \right]^{\frac{1}{2}}$$

Hölder Regularity

A graph signal f is (C, α, r)-Hölder regular with respect to the graph $\mathcal{G}$ at vertex $n \in \mathcal{V}$ if

$$|f(n) - f(m)| \leq C[d_{\mathcal{G}}(m,n)]^{\alpha}, \ \forall m \in \mathcal{N}(n,r)$$

Gavish et al. ICML, 2010

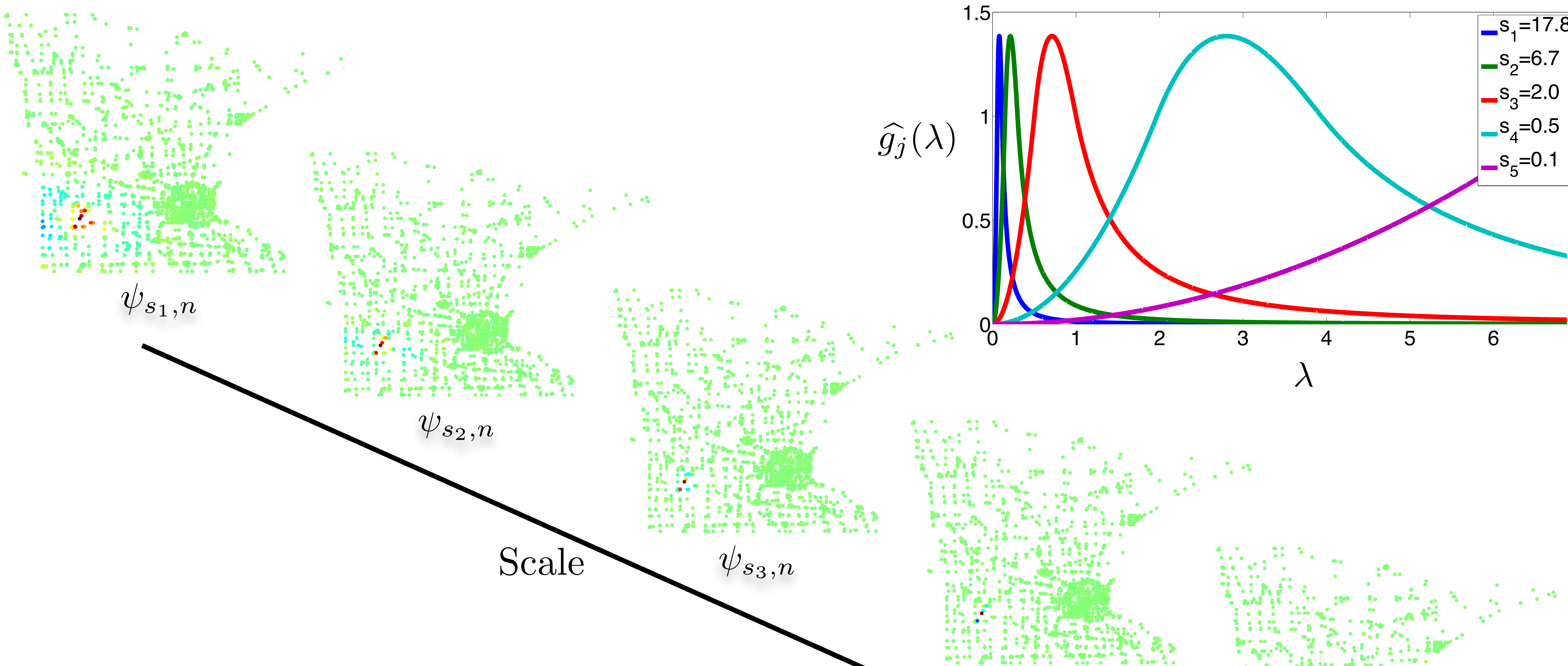
Laplacian as Derivative

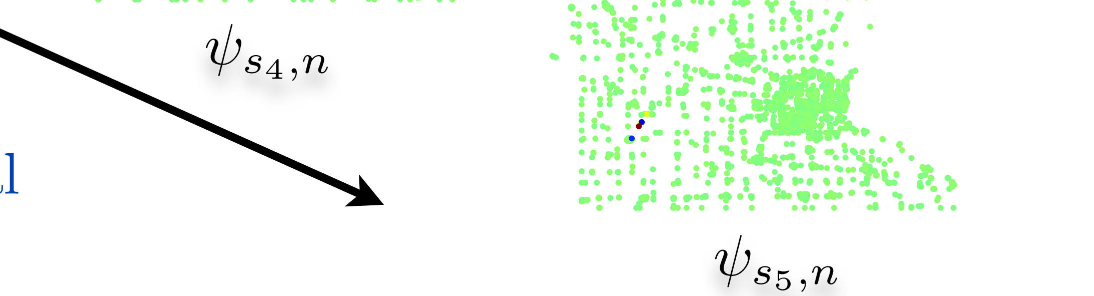
$(\mathcal{L}^k f)(n)$ as a measure of local regularity of f in a neighborhood of radius k around vertex n

- For polynomial kernel:

$$\Psi f(s,n) = \sum_{k=p}^{q} a_k s^k (\mathcal{L}^k f)(n)$$

Spectral Graph Wavelet Localization





Characterizations of this localization

- Hammond et al., Wavelets on graphs via spectral graph theory, 2011
- Shuman et al., Vertex-frequency analysis on graphs, 2013

Wavelet Coefficient Decay of Locally Regular Graph Signals

High-level intuition

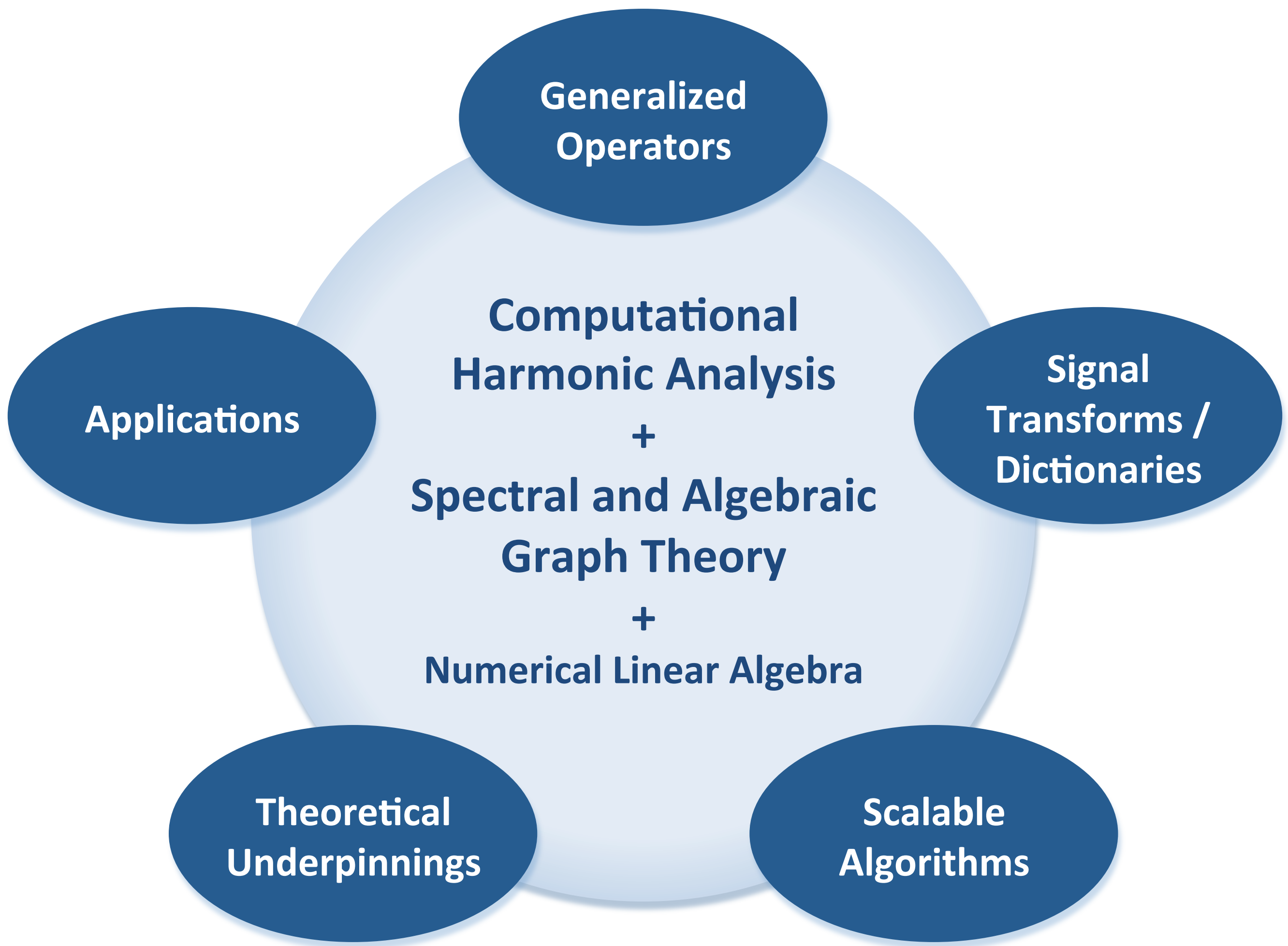
- Far away from vertex n, for small scales s, $|\Psi f(s,n)|$ is small because $\psi_{s,n}$ is highly localized around n
- Close to vertex n, $|\Psi f(s,n)|$ is small because f is locally regular

Proposition 3

Assume that f is (C,α,r)-Hölder regular for some $r \geq 1$, and let $\hat{g}(\lambda) = \sum_{k=r}^{q} a_k \lambda^k$ for some coefficients $\{a_k\}_{k=r,r+1,\ldots,q}$. Then there exist constants C_2 and $\bar{s}$ such that for all $s < \bar{s}$, we have

$$|\Psi f(s,n)| \leq C r^{\alpha} \sum_{m \in \mathcal{N}(n,r)} |\psi_{s,n}(m)| + C_2 s^{r+1} \sum_{m \notin \mathcal{N}(n,r)} |f(m) - f(n)|.$$

Outlook



- Application of graph signal processing techniques to real science and engineering problems is in its infancy
- Theoretical connections between classes of graph signals, the underlying graph structure, and sparsity of transform coefficients