

# Anisotropic adaptive finite elements for evolution problems

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Refinement Strategies for Boundary Element Methods, 23 May 2013

# Adaptive finite elements

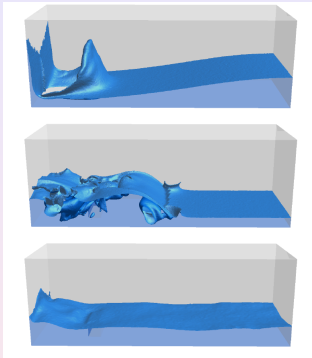
- A posteriori error estimates for elliptic, parabolic, hyperbolic, nonlinear ? problems + efficient meshing tools → adaptive finite elements can be used for industrial problems.
- Adaptive finite element with large aspect ratio: the ultimate tool to reduce the number of vertices given a prescribed level of accuracy.

# Finite elements with large aspect ratio

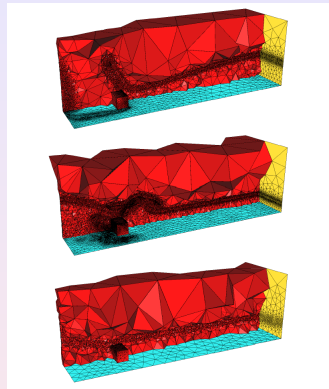
- General statement (mathematician): If nothing is known about the solution, then finite elements with large aspect ratio should not be used.
- A priori error estimates:  $\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch|u|_{H^2(\Omega)}$  and  $C$  is large when the aspect ratio is large.
- But engineers use finite elements with large aspect ratio. Example: viscous compressible flows around aircrafts, aspect ratio  $10^3$  in the boundary layer.
- New statement: finite elements with large aspect ratio can be used provided the mesh fits the solution.
- The theory of finite elements has been recently revisited to handle meshes with large aspect ratio.

- Fluid flows with complex free surfaces (no theory, allows very fast computations).
- Microfluidics (parabolic problem, theory and practice).
- Compressible flows around bodies.

# Example 1: Fluid flows with complex free surfaces



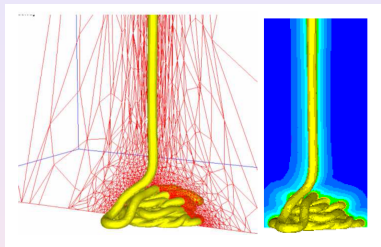
Free surface



Mesh

- Navier-Stokes with level set, finite volumes.
- Anisotropic adaptive remeshing criterion: distance to interface.
- Alain Guégan Alauzet 2009.

# Example 1: Fluid flows with complex free surfaces



Jet buckling of a viscoelastic fluid.

- Viscoelastic fluids with level set, finite elements.
- Anisotropic adaptive remeshing criterion: distance to interface.
- Coupez 2010.

## Example 2: Microfluidics

- Adaptive time steps and finite elements with large aspect ratio.
- Optimal a posteriori error estimates in the  $L^2(H^1)$  norm for the Crank Nicolson scheme on parabolic pde's.
- Lozinski Picasso Prachittham SISC 2009, Picasso Prachittham JCAM 2009, Picasso Prachittham Gijs IJNMF 2009.
- Animation.
- Animation.

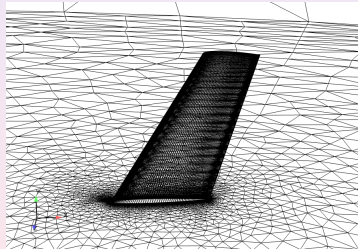
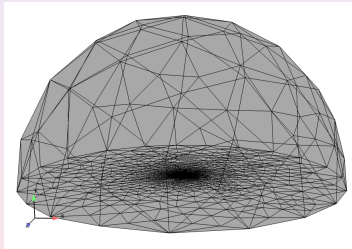
## Example 3: Compressible inviscid flows around bodies

- Bourgault Picasso Alauzet Loseille IJNMF 2009.
- Compressible Euler solver (Alauzet).
- 3D remeshing tool MMG3D (Frey, Dobrzynski).
- Adaptive finite elements with large aspect ratio.
- **Animation.**



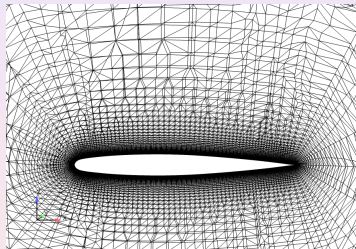
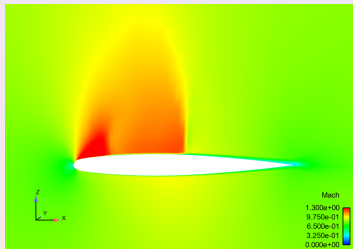
# Example 3: Compressible viscous flows around bodies

- PhD Thesis Wissam Hassan, supported by Dassault Aviation.
- Compressible Navier-Stokes solver (Dassault).
- Adaptive finite elements with large aspect ratio.



# Example 3: Compressible viscous flows around bodies

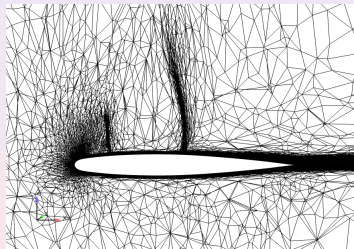
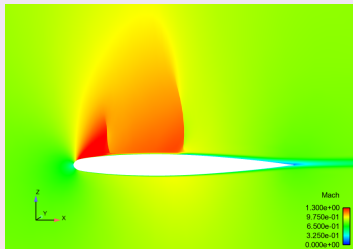
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Mach number for non-adapted meshes.

# Example 3: Compressible viscous flows around bodies

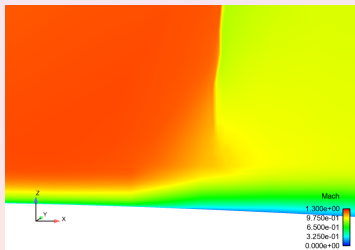
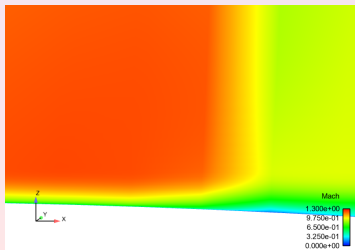
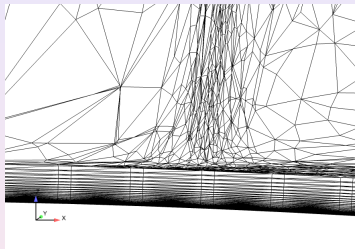
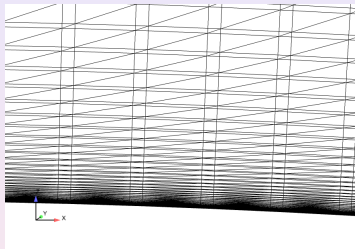
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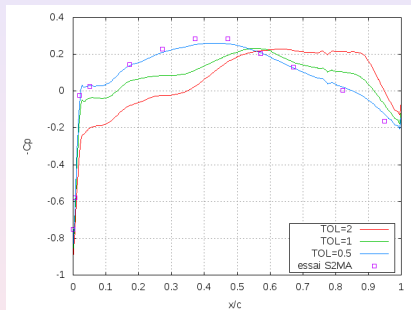
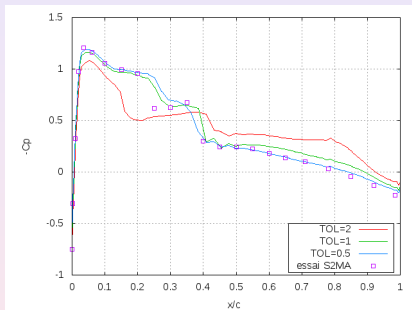
Mach number for adapted meshes.

# Example 3: Compressible viscous flows around bodies

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- Compressible Navier-Stokes solver (Dassault).
- Adaptive finite elements with large aspect ratio.

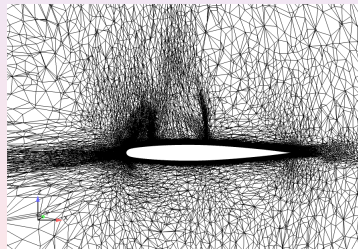
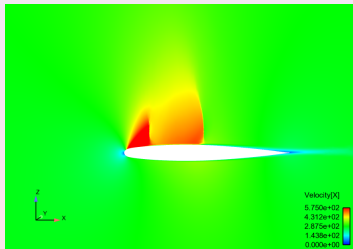
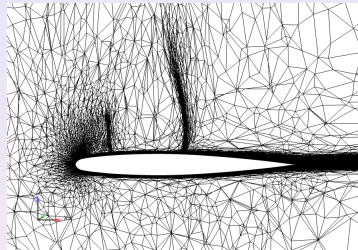
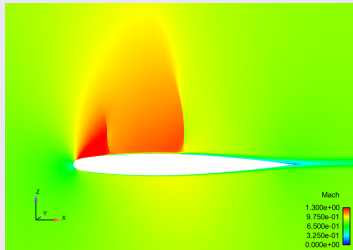


# Example 3: Compressible viscous flows around bodies



Drag along top and bottom of the wing

# Example 3: Compressible viscous flows around bodies



Comparison between  $H^1$  and goal oriented (drag) adaptation

- A priori and a posteriori error estimates (iso and aniso) for the Laplace equation in the natural  $H_0^1(\Omega)$  norm.
- The heat equation in the  $L^2(0, T; H_0^1(\Omega))$  norm.
- The transport equation in the  $C^0([0, T]; L^2(\Omega))$  norm (space semi-discretization only).

# A posteriori error estimates for the Laplace equation

- Find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

- Let  $\mathcal{T}_h$  be a mesh of  $\Omega$  into triangles  $K$  with diameter  $h_K$  less than  $h$ .
- Find  $u_h \in V_h$  (continuous, piecewise linears) such that, for all  $v_h \in V_h$

$$\int_{\Omega} \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h.$$

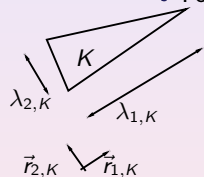


# A priori error estimates for the Laplace equation

- Isotropic meshes:  $\exists C > 0$  (dep. aspect ratio, indep.  $u, h$ )

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch|u|_{H^2(\Omega)}.$$

- Anisotropic meshes: Hessian matrix  $H(u)$ .
  - Dompierre Vallet Bourgault Fortin Habashi 2002, Frey George 2008, Chen Sun Xu 2007, Mirebeau Cohen 2009, Alauzet Loseille 2009.
  - Formaggia Perotto 2001,  $\exists C > 0$  (indep. aspect ratio,  $u, h$ )



$$\int_K |\nabla(u - r_h u)|^2 \leq C \left( \frac{\lambda_{1,K}^4}{\lambda_{2,K}^2} \int_K (\vec{r}_{1,K}^T H(u) \vec{r}_{1,K})^2 + 2\lambda_{1,K}^2 \int_K (\vec{r}_{1,K}^T H(u) \vec{r}_{2,K})^2 + \lambda_{2,K}^2 \int_K (\vec{r}_{2,K}^T H(u) \vec{r}_{2,K})^2 \right).$$

- Ex: the mesh is aligned with  $u$ ,  $u = u(x_2)$ ,  $\vec{r}_{1,K} = (1 \ 0)^T$



$$\int_K |\nabla(u - r_h u)|^2 \leq C \lambda_{2,K}^2 \int_K \left( \frac{\partial^2 u}{\partial x_2^2} \right)^2.$$

- Most anisotropic adaptive algorithms use an estimate of  $H(u)$ , Vallet Manole Dompierre Dufour Guibault 2007, Picasso Alauzet Borouchaki George 2011.

# A posteriori error estimates for the Laplace equation

- Isotropic meshes:  $\exists C_1, C_2 > 0$  (dep. aspect ratio, indep.  $u, h$ )

$$C_1 \eta \leq \|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C_2 \eta + h.o.t.,$$

where the error estimator  $\eta$  is a computable quantity dep. on the mesh size, the data and on  $u_h$ .

- Anisotropic meshes:
  - Kunert, Kunert Verfürth 2000:  $C_2$  depends on the alignment of the mesh with the (unknown) solution  $u$ .
  - Formaggia Perotto 2001 2003, Picasso 2003 2006:  $C_1$  and  $C_2$  indep. aspect ratio,  $u, h$ , provided the error is equidistributed in the directions of min. and max. stretching.

# A posteriori error estimates for the Laplace equation

- The classical procedure to obtain an explicit, residual based error estimator is:

$$\begin{aligned} \int_{\Omega} |\nabla(u - u_h)|^2 &= \int_{\Omega} \nabla(u - u_h) \cdot \nabla(u - u_h) \\ &= \int_{\Omega} f(u - u_h) - \int_{\Omega} \nabla u_h \cdot \nabla(u - u_h) \\ &= \int_{\Omega} f(u - u_h - v_h) - \int_{\Omega} \nabla u_h \cdot \nabla(u - u_h - v_h) \quad \forall v_h \in V_h \\ &= \sum_{K \in \mathcal{T}_h} \left( \int_K (f + \Delta u_h)(u - u_h - v_h) \right. \\ &\quad \left. + \frac{1}{2} \int_{\partial K} [\nabla u_h \cdot n](u - u_h - v_h) \right). \end{aligned}$$

- Use Cauchy-Schwarz inequality, take  $v_h = R_h(u - u_h)$  Clément interpolant, use interpolation estimates to obtain ...

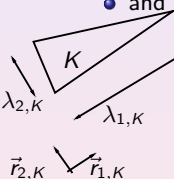
# A posteriori error estimates for the Laplace equation

$$\int_{\Omega} |\nabla(u - u_h)|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K^2,$$

- where, in the isotropic case ( $C$  depends on the aspect ratio)

$$\eta_K^2 = h_K^2 \|f + \Delta u_h\|_{L^2(K)}^2 + \frac{1}{2} |\partial K| \|\nabla u_h \cdot \mathbf{n}\|_{L^2(\partial K)}^2,$$

- and in the anisotropic case ( $C$  does not depend on the aspect ratio)

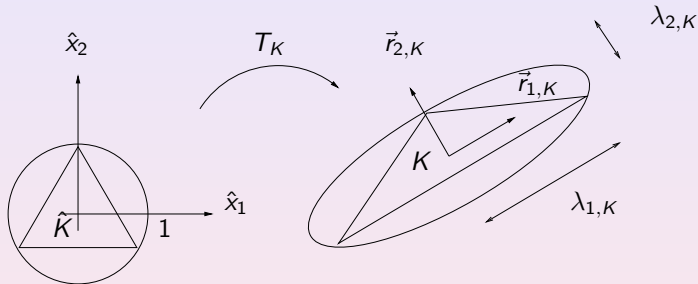

$$\eta_K^2 = \left( \|f + \Delta u_h\|_{L^2(K)} + \frac{1}{2} \left( \frac{|\partial K|}{\lambda_{1,K} \lambda_{2,K}} \right)^{1/2} \|\nabla u_h \cdot \mathbf{n}\|_{L^2(\partial K)} \right) \left( \lambda_{1,K}^2 \left( \vec{r}_{1,K}^T \mathbf{G}_K (u - u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left( \vec{r}_{2,K}^T \mathbf{G}_K (u - u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

- ZZ (Zienkiewicz-Zhu) post-processing to guess  $u - u_h$  (Cao 2013):

$$\mathbf{G}_K(u - u_h) = \begin{pmatrix} \int_{\Delta_K} \left( \frac{\partial(u - u_h)}{\partial x_1} \right)^2 & \int_{\Delta_K} \frac{\partial(u - u_h)}{\partial x_1} \frac{\partial(u - u_h)}{\partial x_2} \\ \int_{\Delta_K} \frac{\partial(u - u_h)}{\partial x_1} \frac{\partial(u - u_h)}{\partial x_2} & \int_{\Delta_K} \left( \frac{\partial(u - u_h)}{\partial x_2} \right)^2 \end{pmatrix}.$$

# Anisotropic interpolation estimates

- Formaggia Perotto, Numer. Math. 2001, 2003.
- See also Kunert, Kunert Verfürth, Numer. Math. 2000.



- $\vec{x} = T_K(\vec{\hat{x}}) = M_K \vec{\hat{x}} + \vec{t}_K$ , s. v. d.  $M_K = R_K^T \Lambda_K P_K$
- The unit circle  $\vec{\hat{x}}^T \vec{\hat{x}} = 1$  is mapped into the ellipse  $(\vec{x} - \vec{t}_K)^T R_K^T \Lambda_K^{-2} R_K (\vec{x} - \vec{t}_K) = 1$ .

# Anisotropic interpolation estimates

- Formaggia Perotto, Numer. Math. 2001, 2003.
- See also Kunert, Kunert Verfürth, Numer. Math. 2000.
- Clément interpolation estimates :  $\exists C > 0$  indep. mesh size and aspect ratio s.t.  $\forall v \in H^1(\Omega), \forall K \in \mathcal{T}_h$  :

$$\begin{aligned} & \|v - R_h v\|_{L^2(K)}^2 + \frac{\lambda_{1,K} \lambda_{2,K}}{|\partial K|} \|v - R_h v\|_{L^2(\partial K)}^2 \\ & \leq C \left( \lambda_{1,K}^2 \left( \vec{r}_{1,K}^T G_K(v) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left( \vec{r}_{2,K}^T G_K(v) \vec{r}_{2,K} \right) \right), \\ & G_K(v) = \begin{pmatrix} \int_{\Delta_K} \left( \frac{\partial v}{\partial x_1} \right)^2 dx & \int_{\Delta_K} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} dx \\ \int_{\Delta_K} \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} dx & \int_{\Delta_K} \left( \frac{\partial v}{\partial x_2} \right)^2 dx \end{pmatrix}. \end{aligned}$$

# A posteriori error estimates for the Laplace equation

$$\int_{\Omega} |\nabla(u - u_h)|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K^2,$$

- Can we prove a lower bound ?
- Yes in the isotropic case, but the constant depends on the aspect ratio (Verfürth 1989).
- Yes in the anisotropic case, the constant does not depend on the aspect ratio provided,  $\forall K \in \mathcal{T}_h$  :

$$\lambda_{1,K}^2 \left( \vec{r}_{1,K}^T G_K (u - u_h) \vec{r}_{1,K} \right) \leq \lambda_{2,K}^2 \left( \vec{r}_{2,K}^T G_K (u - u_h) \vec{r}_{2,K} \right).$$

# Adaptive finite elements

- Goal: find  $\mathcal{T}_h$  such that:

$$0.75 \text{ TOL} \leq \frac{\left( \sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}}{\left( \int_{\Omega} |\nabla u_h|^2 \right)^{1/2}} \leq 1.25 \text{ TOL}.$$

- Sufficient condition ( $N_K$  is the number of triangles):

$$\frac{0.75^2 \text{ TOL}^2 \int_{\Omega} |\nabla u_h|^2}{N_K} \leq \eta_K^2 \leq \frac{1.25^2 \text{ TOL}^2 \int_{\Omega} |\nabla u_h|^2}{N_K}$$

- Isotropic case: if  $\eta_K$  is too large, refine, too small, coarsen.
- Anisotropic case: refine or coarsen in the directions of stretching, align the triangles with the eigenvectors of  $G_K(u - u_h)$ .
- Use the INRIA remeshing tools: BL2D (Laug Borouchaki) GHS (George Hecht Saltel) MMG3D (Dobrzynski Frey).
- Alternative remeshing tools: BAMG (Hecht), Gruau Coupez 2005, Compère Marchandise Remacle 2008.



# Adaptive meshes for the Laplace equation in 2D and 3D

- 2D:  $TOL = 0.25$ , 30 mesh generations, **animation**, **zoom**.
- The effectivity index is aspect ratio independent on adapted meshes

$TOL$	vertices	error	$e_i^{ANI}$	$e_i^{ZZ}$	ar
0.125	854	0.25	2.70	1.00	262
0.0625	2793	0.13	2.75	0.99	288
0.03125	10812	0.062	2.79	0.95	425
0.015625	42562	0.031	2.79	0.98	1199

- 3D:  $TOL = 0.25$ , 30 mesh generations, **animation**, **zoom**.

# The heat equation with Euler backward scheme

- $\frac{\partial u}{\partial t} - \Delta u = f$  in  $\Omega \times (0, T)$ .

- For  $n = 1, \dots, N$ , find  $u_h^n \in V_h$  such that, for all  $v_h \in V_h$

$$\frac{1}{\tau} \int_{\Omega} (u_h^n - u_h^{n-1}) v_h dx + \int_{\Omega} \nabla u_h^n \cdot \nabla v_h dx = \int_{\Omega} f^n v_h v dx.$$

- Isotropic meshes: Picasso 1998, Verfürth 2003, Bergam Bernardi Mghazli 2004.

- $u_{h\tau}(x, t) = \frac{t - t^{n-1}}{\tau} u_h^n(x) + \frac{t^n - t}{\tau} u_h^{n-1}(x).$

- $$\int_{\Omega} \frac{\partial u_{h\tau}}{\partial t} v_h dx + \int_{\Omega} \nabla u_{h\tau} \cdot \nabla v_h dx$$
$$= \int_{\Omega} f v_h dx + \int_{\Omega} (f^n - f) v_h dx + \int_{\Omega} \nabla (u_{h\tau} - u_h^n) \cdot \nabla v_h dx.$$

# The heat equation with Euler backward scheme

- $e = u - u_{h\tau}$

$$\begin{aligned} \left\langle \frac{\partial e}{\partial t}, e \right\rangle + \int_{\Omega} |\nabla e|^2 dx &= \int_{\Omega} \left( f - \frac{\partial u_{h\tau}}{\partial t} \right) e dx - \int_{\Omega} \nabla u_{h\tau} \cdot \nabla e dx \\ &= \int_{\Omega} \left( f - \frac{\partial u_{h\tau}}{\partial t} \right) (e - v_h) dx - \int_{\Omega} \nabla u_{h\tau} \cdot \nabla (e - v_h) dx \\ &\quad + \int_{\Omega} (f - f^n) v_h dx + \int_{\Omega} \nabla (u_h^n - u_{h\tau}) \cdot \nabla v_h dx. \end{aligned}$$

- Take  $v_h = R_h e$ , use anisotropic interpolation estimates:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (u - u_{h\tau})^2(T) dx + \int_0^T \int_{\Omega} |\nabla (u - u_{h\tau})(t)|^2 dx dt \\ \leq \frac{1}{2} \int_{\Omega} (u - u_{h\tau})^2(0) dx + C \sum_{n=1}^N \sum_{K \in \mathcal{T}_h} \eta_{K,n}^2. \end{aligned}$$

# The heat equation with Euler backward scheme

- with  $C$  independent of  $u$ , the mesh size and aspect ratio and

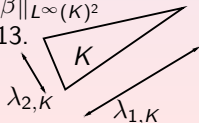
$$\begin{aligned} & \eta_{K,n}^2 \\ &= \int_{t^{n-1}}^{t^n} \left\{ \left( \left\| f - \frac{\partial u_{h\tau}}{\partial t} + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \|\nabla u_{h\tau} \cdot \vec{n}\|_{L^2(\partial K)} \right) \right. \\ & \quad \left. \left( \lambda_{1,K}^2 \left( \vec{r}_{1,K}^T G_K (u - u_{h\tau}) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left( \vec{r}_{2,K}^T G_K (u - u_{h\tau}) \vec{r}_{2,K} \right) \right)^{1/2} \right. \\ & \quad \left. + \|f - f^n\|_{L^2(K)}^2 + \|\nabla(u_{h\tau} - u_h^n)\|_{L^2(K)}^2 \right\} dt. \end{aligned}$$

# A posteriori error estimates for a finite element discretization of the transport equation

- Isotropic case: Diallo Pousin Sassi 1999, Houston Rannacher Suli 2000.
- $\frac{\partial u}{\partial t} + \beta \cdot \nabla u = f$  in  $\Omega \times (0, T)$ ,
- $u = 0$  on  $\Gamma^- = \{x \in \partial\Omega; \beta \cdot n < 0\}$ .
- Here  $\beta \in C^1(\bar{\Omega})$ ,  $\operatorname{div} \beta = 0$  in  $\Omega$ .
- Find  $u_h : t \rightarrow u_h(\cdot, t) \in V_h$  such that, for all  $t \in (0, T)$ , for all  $v_h \in V_h$

$$\int_{\Omega} \left( \frac{\partial u_h}{\partial t} + \beta \cdot \nabla u_h - f \right) \left( v_h + \delta_h \beta \cdot \nabla v_h \right) dx = 0.$$

- Isotropic case,  $\delta_{h|K} = \frac{h_K}{2\|\beta\|_{L^\infty(K)^2}}$ , Burman 2010.
- Anisotropic case,  $\delta_{h|K} = \frac{\lambda_{2,K}}{2\|\beta\|_{L^\infty(K)^2}}$ , Micheletti Perotto Picasso 2003, Bourgault Picasso 2013.



# Error estimates for a finite element discretization of the transport equation

- Find  $u_h : t \rightarrow u_h(\cdot, t) \in V_h$  such that, for all  $t \in (0, T)$

$$\int_{\Omega} \left( \frac{\partial u_h}{\partial t} + \beta \cdot \nabla u_h - f \right) \left( v_h + \delta_h \beta \cdot \nabla v_h \right) dx = 0 \quad \forall v_h \in V_h.$$

- Stability (as in Burman 2010): set  $f = 0$ , take a constant  $\delta_h$ , choose  $v_h = u_h + \delta_h \partial u_h / \partial t$  to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_h^2 dx + \int_{\Omega} \delta_h \left( \frac{\partial u_h}{\partial t} + \beta \cdot \nabla u_h \right)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \delta_h^2 (\beta \cdot \nabla u_h)^2 dx \leq 0.$$

- Anisotropic, a priori error estimates, in the isotropic case

$$\int_{\Omega} (u - u_h)^2(T) dx \leq Ch^3.$$

# A posteriori error estimates for a finite element discretization of the transport equation

- $\frac{\partial u}{\partial t} + \beta \cdot \nabla u = f$  in  $\Omega \times (0, T)$ .
- $e = u - u_h$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} e^2 dx &\leq \int_{\Omega} \left( \frac{\partial e}{\partial t} e + (\beta \cdot \nabla e) e \right) dx \\ &= \int_{\Omega} \left( f - \frac{\partial u_h}{\partial t} - \beta \cdot \nabla u_h \right) e dx \\ &= \int_{\Omega} \left( f - \frac{\partial u_h}{\partial t} - \beta \cdot \nabla u_h \right) \left( e - v_h - \delta_h \beta \cdot \nabla v_h \right) dx, \end{aligned}$$

for all  $v_h \in V_h$ .

- Use Cauchy-Schwarz inequality, take  $v_h = R_h e$  Clément interpolant, use interpolation estimates to obtain ...

# A posteriori error estimates for a finite element discretization of the transport equation

$$\int_{\Omega} (u - u_h)^2(T) dx \leq \int_{\Omega} (u - u_h)^2(0) dx + C \int_0^T \sum_{K \in \mathcal{T}_h} \eta_K^2,$$

where

$$\eta_K^2 = \left\| f - \frac{\partial u_h}{\partial t} - \beta \cdot \nabla u_h \right\|_{L^2(K)}^2 \left( \lambda_{1,K}^2 \left( \vec{r}_{1,K}^T G_K(u - u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left( \vec{r}_{2,K}^T G_K(u - u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

ZZ (Zienkiewicz-Zhu) post-processing to guess  $G_K(u - u_h)$ :

$$G_K(u - u_h) = \begin{pmatrix} \int_{\Delta_K} \left( \frac{\partial(u - u_h)}{\partial x_1} \right)^2 & \int_{\Delta_K} \frac{\partial(u - u_h)}{\partial x_1} \frac{\partial(u - u_h)}{\partial x_2} \\ \int_{\Delta_K} \frac{\partial(u - u_h)}{\partial x_1} \frac{\partial(u - u_h)}{\partial x_2} & \int_{\Delta_K} \left( \frac{\partial(u - u_h)}{\partial x_2} \right)^2 \end{pmatrix}.$$



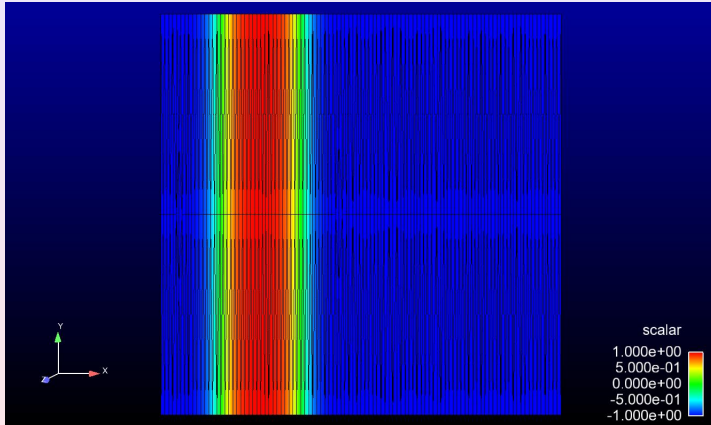
- Crank-Nicolson scheme with time step  $\tau$ :

$$\int_{\Omega} \left( \frac{u_h^{n+1} - u_h^n}{\tau} + \frac{1}{2} \beta \cdot \nabla (u_h^{n+1} + u_h^n) - \frac{1}{2} (f(t^{n+1}) + f(t^n)) \right) \left( v_h + \delta_h \beta \cdot \nabla v_h \right) dx = 0 \quad \forall v_h \in V_h,$$

- Introduce  $u_{h\tau}(x, t) = \frac{t - t^n}{\tau} u_h^{n+1}(x) + \frac{t^{n+1} - t}{\tau} u_h^n(x)$ .
- The error  $(\int_{\Omega} (u - u_{h\tau})^2(T) dx)^{1/2} = O(h^{3/2} + \tau^2)$ .
- Therefore when  $\tau = O(h^2)$ , the error due to time discretization should be negligible.

# Numerical results for unstructured, non adapted, anisotropic meshes

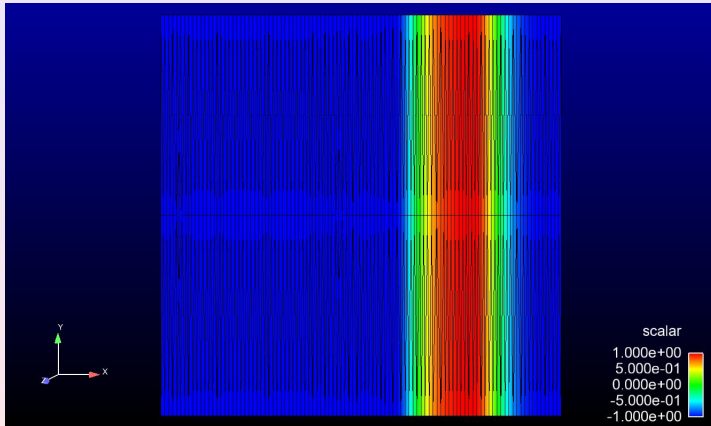
Aspect ratio 50



Initial time

# Numerical results for unstructured, non adapted, anisotropic meshes

Aspect ratio 50



Final time

# Numerical results for unstructured, non adapted, anisotropic meshes

$h$	$\tau$	$e_{L^2(H^1)}$	$ei^{ZZ}$	$e(T)_{L^2}$	$ei^{ANI}$	$ar$
0.01-0.5	0.002	0.13	1.00	0.0013	14.3	50
0.005-0.25	0.0005	0.067	1.00	0.00047	15.7	50
0.0025-0.125	0.000125	0.034	1.00	0.00015	17.2	50
0.00125-0.0625	0.00003125	0.017	1.00	0.000042	20.1	50
0.0005-0.25	0.00003125	0.0062	1.01	0.000011	20.7	500

# Adaptive finite elements

- Goal: to control  $(\int_{\Omega} (u - u_{h\tau})^2(T) dx)^{1/2}$

$$0.75 \text{ TOL} \leq \left( \frac{\sum_{K \in \mathcal{T}_h} \int_0^T \eta_K^2}{T} \right)^{1/2} \leq 1.25 \text{ TOL}.$$

- Sufficient condition, for each  $n = 0, 1, 2, \dots, N - 1$ , build a triangulation  $\mathcal{T}_h^n$  such that

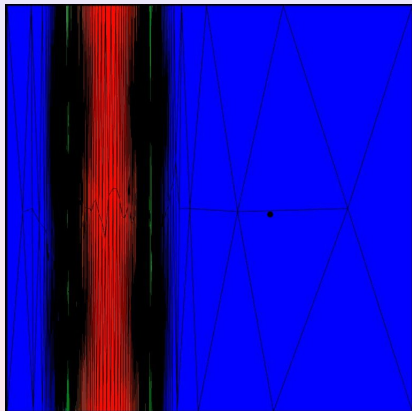
$$0.75^2 \text{ TOL}^2 \tau \leq \sum_{K \in \mathcal{T}_h} \int_{t^n}^{t^{n+1}} \eta_K^2 \leq 1.25^2 \text{ TOL}^2 \tau.$$

- Equidistribute  $\eta_K$  ( $N_K$  is the number of triangles)

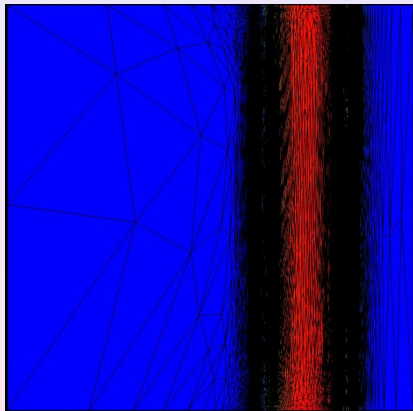
$$\frac{0.75^2 \text{ TOL}^2 \tau}{N_K} \leq \int_{t^n}^{t^{n+1}} \eta_K^2 \leq \frac{1.25^2 \text{ TOL}^2 \tau}{N_K}.$$

- Isotropic case: if  $\eta_K$  is too large, refine, too small, coarsen.
- Anisotropic case: refine or coarsen in the directions of stretching, align the triangles with the eigenvectors of  $G_K(u - u_h)$ .

# Numerical results for adapted, anisotropic meshes



Initial time



Final time

# Numerical results for adapted, anisotropic meshes

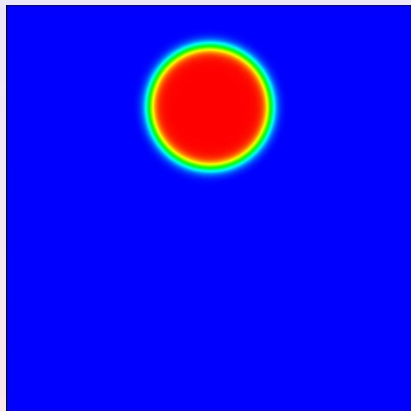
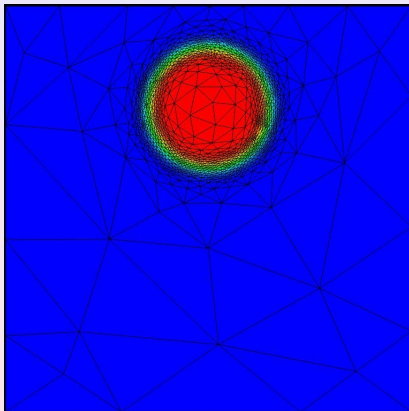
$TOL$	$\tau$	$e_{L^2(H^1)}$	$ei_{ZZ}$	$e(T)_{L^2}$	$ei_{ANI}$	$N_v$	$ar$	$N_{mesh}$
0.01	0.002	0.075	0.82	0.0035	1.95	850	65	29
0.005	0.0005	0.044	0.89	0.0015	2.35	1689	82	31
0.0025	0.000125	0.026	0.95	0.00062	2.82	2421	129	42
0.00125	0.00003125	0.025	0.63	0.00089	0.53	9391	98	50

Linear interpolation between meshes

$TOL$	$\tau$	$e_{L^2(H^1)}$	$ei_{ZZ}$	$e(T)_{L^2}$	$ei_{ANI}$	$N_v$	$ar$	$N_{mesh}$
0.01	0.002	0.063	0.98	0.00068	10.12	715	87	37
0.005	0.0005	0.039	0.99	0.00023	14.66	1187	104	41
0.0025	0.000125	0.024	1.00	0.00012	14.71	2856	119	43
0.00125	0.00003125	0.015	1.00	0.000051	17.25	5659	149	44

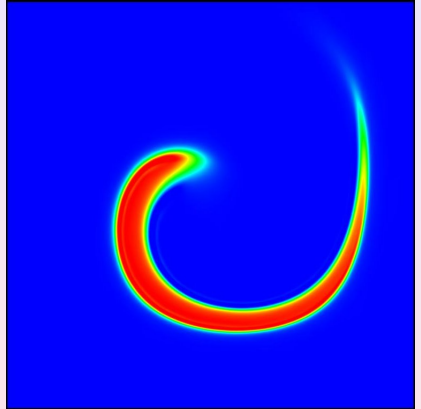
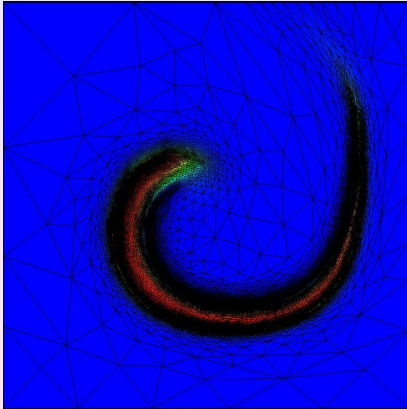
Conservative interpolation between meshes (Alauzet Mehrenberger 2010)

# Numerical results for adapted, anisotropic meshes

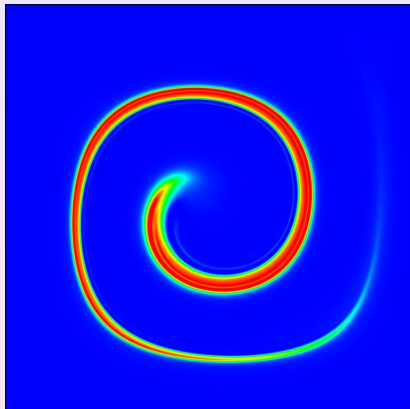
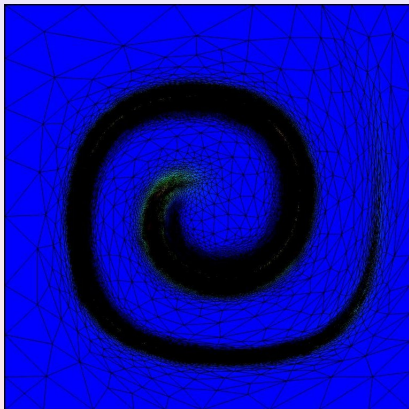




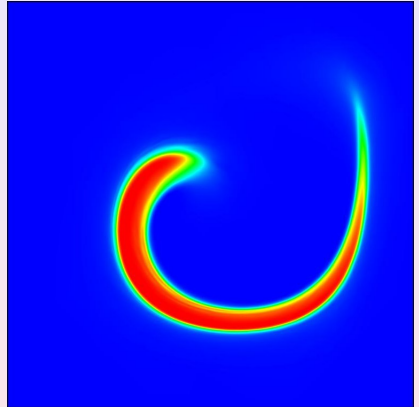
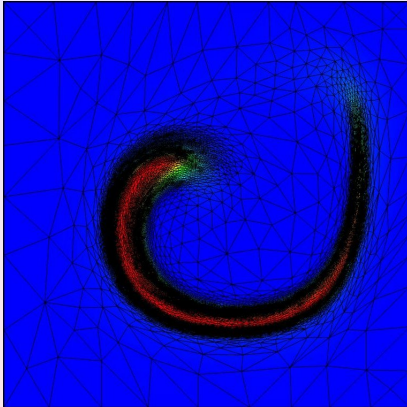
# Numerical results for adapted, anisotropic meshes



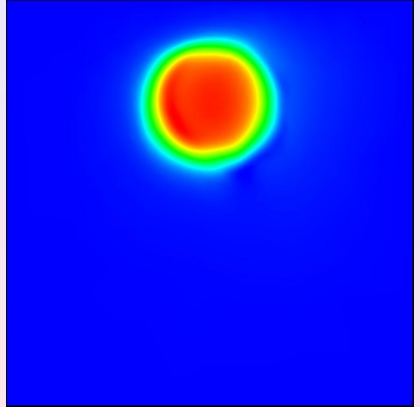
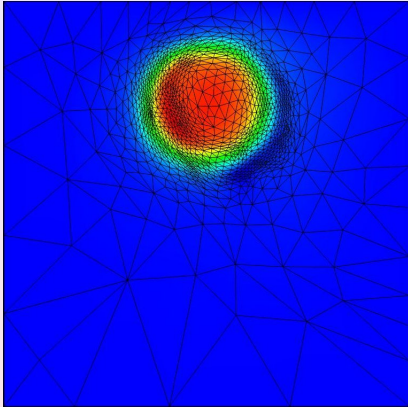
# Numerical results for adapted, anisotropic meshes



# Numerical results for adapted, anisotropic meshes



# Numerical results for adapted, anisotropic meshes



# Conclusions

- Adaptive finite elements with large aspect ratio → reduce the number of vertices.
- Anisotropic interpolation estimates (Formaggia-Perotto, Kunert) + ZZ postprocessing → residual based, explicit anisotropic error estimator for the  $H^1$  norm.
- Effectivity index aspect ratio independent whenever the estimator is equidistributed in the direction of maximum and minimum stretching.
- Applied to a wide range of elliptic, parabolic and hyperbolic problems, very well suited for problems with internal or boundary layers.

- Nonlinear (engineering) problems? Compressible Navier-Stokes?  
Incompressible Navier-Stokes with free surfaces?
- Interpolation error induced by remeshing?
- Intersection between meshes.