1 Transversal eigenmodes in the slit array

This chapter details the calculation of TE and TM eigenmodes in the slit array. The vectorial time independent Helmholtz equation is applyied to our model geometry. The numerical eigenvalue problem is established and solved as outlined in [1].

1.1 Analytical formulation for our model geometry

According to equations (9[1]) and (10[1]), we seek the transversal eigenmodes of the slit array in terms of

$$\vec{E}_{\vec{k}_{/\!/},l(\vec{r})} = \vec{E}_{\gamma_0,l(x)} e^{ik_0(\gamma_0 x+\beta_l z)} \qquad \text{for TE polarisation} \tag{1a}$$

and

$$\vec{H}_{\vec{k}_{/\!/},l(\vec{r})} = \vec{H}_{\gamma_0,l(x)} e^{ik_0(\gamma_0 x+\beta_l z)} \qquad \text{for TM polarisation.} \tag{1b}$$

The eigenmode has a periodic amplitude $\vec{U}_{\gamma_0,l(x)} = \vec{U}_{\gamma_0,l(x+mp_x)}\ \forall\ m \in \mathbb{Z}$ independent of the propagation term $e^{ik_0(\gamma_0 x+\beta_l z)}$. It is characterised by $\gamma_0 = k_x/k_0 = \sqrt{\epsilon_1}\sin\theta_i$ where θ_i is the incidence angle and a mode number $l \in \mathbb{N}$.

The insertion in equations (6[1]) and (7[1]) yields the vectorial time independent Helmholtz equation for the slit array.

$$\frac{1}{\epsilon_{(x)}}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}_{\gamma_0,l(x)}e^{ik_0(\gamma_0 x+\beta_l z)}\right) = k_0^2\vec{E}_{\gamma_0,l(x)}e^{ik_0(\gamma_0 x+\beta_l z)} \qquad \text{for TE polarisation} \tag{2a}$$

$$\vec{\nabla}\times\left(\frac{1}{\epsilon_{(x)}}\vec{\nabla}\times\vec{H}_{\gamma_0,l(x)}e^{ik_0(\gamma_0 x+\beta_l z)}\right) = k_0^2\vec{H}_{\gamma_0,l(x)}e^{ik_0(\gamma_0 x+\beta_l z)} \qquad \text{for TM polarisation} \tag{2b}$$

where

$$\epsilon_{(x)} = \begin{cases} \epsilon_2 & \forall \quad \frac{w_x}{2} < |x| < \frac{p_x}{2} \\ \epsilon_3 & \forall \quad |x| < \frac{w_x}{2} \end{cases} \qquad \text{is the relative dielectric constant in the slit array.}$$

Next, we rewrite the vectorial equation (2b) for the single non-null y-component $H_{y(x)}$.

$$\begin{aligned}
&\vec{\nabla}\times\left(\frac{1}{\epsilon_{(x)}}\begin{pmatrix} -ik_0\beta_l H_y \\ 0 \\ ik_0\gamma_0 H_y + \frac{dH_y}{dx}\end{pmatrix} e^{ik_0(\gamma_0 x+\beta_l z)}\right) \\
&= \begin{pmatrix} 0 \\ \frac{1}{\epsilon}k_0^2\beta_l^2 H_y + \frac{1}{\epsilon^2}\frac{d\epsilon}{dx}\left(ik_0\gamma_0 H_y + \frac{dH_y}{dx}\right) - \frac{1}{\epsilon}\left(\frac{d^2H_y}{dx^2} + 2ik_0\gamma_0\frac{dH_y}{dx} - k_0^2\gamma_0^2 H_y\right) \\ 0 \end{pmatrix} e^{ik_0(\gamma_0 x+\beta_l z)} \\
&= k_0^2\begin{pmatrix} 0 \\ H_y \\ 0 \end{pmatrix} e^{ik_0(\gamma_0 x+\beta_l z)}
\end{aligned} \tag{3}$$

For TM polarisation, the y-component yields

$$\left(\epsilon k_0^2 - ik_0\gamma_0\frac{1}{\epsilon}\frac{d\epsilon}{dx}\right)H_y + \left(2ik_0\gamma_0 - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\right)\frac{dH_y}{dx} + \frac{d^2H_y}{dx^2} = \epsilon_l k_0^2 H_y \tag{4a}$$

and for TE polarisation

$$\epsilon k_0^2 E_y + 2ik_0\gamma_0\frac{dE_y}{dx} + \frac{d^2E_y}{dx^2} = \epsilon_l k_0^2 E_y \tag{4b}$$

where $\epsilon_l = \gamma_0^2 + \beta_l^2$ is the effective dielectric constant of the l^{th} eigenmode.

1.2 Numerical evaluation − eigenvalue problem

We are going to sample the equation (4a) at N points a period p_x. Hence, we choose

$$x_m = m\Delta x = m\frac{p_x}{N} \qquad \text{as sampling points}$$

and abreviate

$$H_m = H_{y(x_m)} \qquad \text{as well as} \qquad \epsilon_m = \epsilon_{(x_m)}$$

Approximating the first derivatives by

$$\frac{d\epsilon}{dx} \approx \frac{\epsilon_{m+1} - \epsilon_{m-1}}{2\Delta x} \qquad \text{and} \qquad \frac{dH_y}{dx} \approx \frac{H_{m+1} - H_{m-1}}{2\Delta x}$$

and the second derivative by

$$\frac{d^2H_y}{dx^2} \approx \frac{H_{m+1} - 2H_m + H_{m-1}}{(\Delta x)^2}$$

we get the sampled version of equation (4a)

$$a_m H_m + b_m (H_{m+1} - H_{m-1}) + c (H_{m+1} - 2H_m + H_{m-1}) = \epsilon_l k_0^2 H_m \tag{5}$$

with the coefficients

$$\begin{aligned} a_m &= \epsilon_m k_0^2 - ik_0\gamma_0 \frac{1}{\epsilon_m} \frac{\epsilon_{m+1} - \epsilon_{m-1}}{2\Delta x} \\ b_m &= \frac{1}{2\Delta x}\left(2ik_0\gamma_0 - \frac{1}{\epsilon_m}\frac{\epsilon_{m+1} - \epsilon_{m-1}}{2\Delta x}\right) \\ c &= \frac{1}{(\Delta x)^2} \end{aligned}$$

Equation (5) has to hold at every sampling point. Hence, we rewrite the N equations in matrix form as

$$\begin{pmatrix} a_1 - 2c & b_1 + c & 0 & \dots & 0 & c - b_1 \\ c - b_2 & a_2 - 2c & b_1 + c & & & 0 \\ 0 & c - b_3 & a_3 - 2c & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & b_{N-2} + c & 0 \\ 0 & & & c - b_{N-1} & a_{N-1} - 2c & b_{N-1} + c \\ b_N + c & 0 & \dots & 0 & c - b_N & a_N - 2c \end{pmatrix} \begin{pmatrix} H_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ H_N \end{pmatrix} = \epsilon_l k_0^2 \begin{pmatrix} H_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ H_N \end{pmatrix} \tag{6}$$

where we used $H_0 = H_N$ and $H_{N+1} = H_1$ due to the periodicity along the x-axis. Equation (6) is a classic eigenvalue equation of the form

$$M \cdot \vec{H}_{\{m\}} = \epsilon_l k_0^2 \vec{H}_{\{m\}}$$

where

$\epsilon_l k_0^2$ is an eigenvalue of the matrix M and

$\vec{H}_{\{m\}}$ is the corresponding $N \times 1$ eigenvector.

In an analog manner, we sample equation (4b) and get for TE polarisation

$$\begin{pmatrix} a_1-2c & b+c & 0 & \dots & 0 & c-b \\ c-b & a_2-2c & b+c & & & 0 \\ 0 & c-b & a_3-2c & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & b+c & 0 \\ 0 & & & c-b & a_{N-1}-2c & b+c \\ b+c & 0 & \dots & 0 & c-b & a_N-2c \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ \vdots \\ E_N \end{pmatrix} = \epsilon_l k_0^2 \begin{pmatrix} E_1 \\ \vdots \\ \vdots \\ E_N \end{pmatrix} \tag{7}$$

where

$$a_m = \epsilon_m k_0^2 \qquad b = \frac{ik_0\gamma_0}{\Delta x} \qquad \text{and} \qquad E_m = E_{y(x_m)}$$

1.3 Stability of the numerical evaluation

In general, choosing a high number N improoves the accuracy of the numerical result. Unfortunately, above a critical N_c, the algorithm may turn instable and produce a modulated result.

Due to the derivations, high frequency fluctuations [1] are amplified. Indeed, we noted that for **even** N, the eigenmodes $\vec{H}_{\{m\}}$ and $\vec{E}_{\{m\}}$ tend to oscillate at the sampling frequency. Nevertheless, the signal corresponds to the enveloppe and can be retrieved by filtering.

Inspecting equation (3), we find the major origin of the oscillation in the second derivative of H_y. This derivative is a high pass filter that is the most sensitive at the sampling frequency $f_x = \frac{N}{2p_x}$, hence boosting the sampling noise. An initial perturbation at this frequency propagates through the row equations in (6) or (7) and leads to a modulation of the signal we looked for. Although, we should note that for **odd** N, the high frequency oscillation is efficiently suppressed [2].

Conclusions:

- The algorithm is **stable for odd** N and for even $N < N_c$.
- The result converges for increasing N. In particular, the accuracy of the eigenmodes $\vec{H}_{\{m\}}$ and $\vec{E}_{\{m\}}$ profit of higher N whereas the effective dielectric constant ϵ_l is less affected.

[1] Sampling and quantisation noise for example.

[2] In fact, it creates a negative retroaction earesing its own origins.

References

[1] M. Leutenegger, *Computation of custom made photonic crystals*, Semester work at the EPFL, Monitors: MER. Dr. P. Hoffmann and Prof. O. Martin (ETHZ), Lausanne (2002)