Quantum Effects in Leptogenesis
In the Case of Quasi-Degenerate Majorana Masses

Christopher Andrey

Dr. Alexey Anisimov
Adviser

Prof. Mikhail Shaposhnikov
LPPC
Swiss Federal Institute of Technology Lausanne (EPFL)

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A. Majorana Fields
B. Traces computation

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INTRODUCTION

The cosmological data at present provide accurate measurements of matter-antimatter asymmetry with \( \frac{n_b}{n_\gamma} = (6.3 \pm 0.3) \times 10^{-10} \). It is known that Electroweak Baryogenesis within Standard Model (SM) of particle interactions may not generate such an asymmetry given current lower bound on the Higgs mass. On the other hand, the observed active neutrino mixings cannot be incorporated within the Standard Model and, as well, require the departure from the known particle content and interaction picture. A great variety of extensions of the Standard Model were proposed over the past years to address these issues. The Seesaw extensions are by far the simplest extensions of the Standard Model which incorporate both the natural explanation for the active neutrino mass scale and mixings at the same time yielding a mechanism for the generation of the asymmetry in the light neutrinos, leptogenesis, which is partially converted into baryon asymmetry via sphalerons effects.

The results from neutrino mixing experiments represent a success for the see-saw mechanism. The atmospheric and the solar neutrino mass scales point indeed to a very plausible grand unified scale of \( \sim 10^{15} \text{ GeV} \) and the observed large mixing angles can be understood within many well motivated see-saw models. At the same time, neutrino mixing data represent a positive test for leptogenesis, an attractive explanation of the observed matter-antimatter asymmetry of the Universe and a cosmological direct consequence of the see-saw mechanism.

Despite great progress during last years in deriving interesting constraints on see-saw parameters, we still do not have a way to probe leptogenesis and the see-saw mechanism directly. The main obstacle is that the heavy neutrinos cannot be produced in accelerators, at least not within typical pictures, where the right-handed (RH) neutrinos are much heavier than TeV or too weakly coupled. Moreover, after producing the asymmetry, they typically decay very fast thus disappearing from the cosmological lore. In the standard leptogenesis scenario with hierarchical RH neutrino mass spectrum it is known that the observed matter-antimatter asymmetry can be produced provided the lightest RH neutrino has mass larger than about \( 10^8 \text{ GeV} \). This is because the CP asymmetry in the decay of the lightest RH neutrino which comes from two types of diagrams, triangle diagram and one loop correction to the RH neutrino self-energy diagram, is roughly proportional to its mass and becomes too small if the mass go below \( 10^8 \text{ GeV} \). However, as it was noted in [PU04, ABP06] the CP-asymmetry coming from the self-energy correction, in the case when at
least two lightest RH neutrinos are degenerate in mass, is enhanced and the lightest RH neutrino mass scale can be pulled down to TeV scale. This possibility has potentially interesting applications in search for some certain rare decays [PU05].

The evolution of RH neutrino and the asymmetry in the light lepton sector were studied in [PU04] by the means of Boltzmann equations. However, in the case of degenerate RH neutrinos flavor oscillations may have a significant effect on the evolution of the resulting asymmetry in the light lepton sector. The aim of this diploma work is to study the effect of that by the means of the evolution of the neutrino density matrix. The quantum kinetic equation (QKE) [SR93, ARS98, AS05] is the basis for this study.
NOTATIONS

The notation of this work will use the $\hbar = c = k_B = 1$ convention.

Furthermore we list here some specific notations we will use.

Bose-Einstein and Fermi-Dirac phase space distribution will be written $n_B$ and $n_F$, $n$ in the Boltzmann approximation.

Bose-Einstein and Fermi-Dirac number density will generically be written $N_F$ and $N_B$. When writing the Boltzmann equations we will use the symbol of the particle. For example $\nu_{L_2}$ will stand for $n_F(\nu_{L_2})$. Remember that

$$N_{B/F} = g_a \int \frac{d^3p}{(2\pi)^3} n_{B/F}$$

with $g_a$ the internal d.o.f.

The letter $z$ will stand for the ratio of the mass of a particle over temperature

$$z_i = \frac{m_i}{T}$$

The expression $\Delta X$ will always be defined as $X_1 - X_2$ regardless of the nature of $X$.

As we are often be dealing with degenerate Majorana neutrinos the following symbols will always be related to them:

<table>
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<tr>
<th>Physical quantity</th>
<th>Symbol</th>
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<tr>
<td>Mass</td>
<td>$M$</td>
</tr>
<tr>
<td>Momentum</td>
<td>$k$</td>
</tr>
<tr>
<td>Energy</td>
<td>$\omega$</td>
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<tr>
<td>Mass over temperature ratio</td>
<td>$z$</td>
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Here the mass scale is defined as $M = M_1 \simeq M_2$. 
1. THE SCENARIO

1.1 General Framework

The Lagrangian we are going to use is the see-saw extension of the Standard Model Lagrangian $L_{SM}$. The Lagrangian has the following form

$$
L = L_{SM} + i \bar{N}_i \frac{\partial}{\partial N_i} - F^{ik} \overline{L}_i N_k \Phi^c - \frac{M_i}{2} \overline{N}_i N_i + \text{h.c.} \quad (1.1)
$$

The first extra term is the kinetic term of the Majorana neutrinos. The second one contains the Dirac mass terms and the interactions of the Majorana neutrinos with light neutrinos and the Higgs field. The third term is the Majorana mass term. The $F$ matrix contains Yukawa couplings which order is constrained by the active neutrino masses.

1.2 See-saw Mechanism

The constraints on Yukawa couplings arise after one diagonalises the mass terms of the Lagrangian $L$ after electroweak symmetry is broken. With the assumption that the Majorana mass is much larger than the Dirac mass the light neutrino masses $O(10^{-2}\text{eV})$ can be generated. After the electroweak symmetry is broken one of the component of the Higgs field acquires expectation value (vev):

$$
\Phi \rightarrow \left( \begin{array}{c} 0 \\ v + \frac{\phi}{\sqrt{2}} \end{array} \right) \quad (1.2)
$$

where $v = 174$ GeV. Then the mass term in the Lagrangian (1.1) can be written as

$$
-\mathcal{L}_{\text{mass}} = v F^{ik} \overline{\nu}_L N_k + \frac{M_i}{2} \overline{N}_i N_i \equiv \overline{\nu}_L M_D N + \frac{1}{2} \overline{N}_i M N + \text{h.c.} \quad (1.3)
$$

Taking into account that $\overline{N}_i M_D \nu_L = \overline{\nu}_L M_D^T N$ one can rewrite the previous expression as

$$
\frac{1}{2} \left( \overline{\nu}_L, \overline{N}_i \right) \left( \begin{array}{cc} 0 & M_D \\ M_D^T & M \end{array} \right) \left( \begin{array}{c} \nu_L \\ N \end{array} \right) \quad (1.4)
$$

The mass matrix is first rotated to the block-diagonal form with the matrix $U$ as

$$
U \left( \begin{array}{cc} 0 & M_D \\ M_D^T & M \end{array} \right) U^T = \left( \begin{array}{cc} -M_\nu & 0 \\ 0 & M \end{array} \right) \quad (1.5)
$$
To the first order in $\frac{MD}{M} \ll 1$ the $U$ matrix is given by

$$U = \begin{pmatrix} 1 & \theta \\ -\theta^T & 1 \end{pmatrix}$$

which satisfies $U^T U = UU^T = 1$. The angle $\theta^1$ and the active neutrino mass matrix are given by

$$\theta = -M_D M^{-1}, \quad M_\nu = M_D M^{-1} M_D^T$$

where we have neglected terms of $O(\theta^2 M_D)$ and used the fact that $M$ is diagonal. Diagonalising further the active neutrino mass matrix allows to relate its eigenvalues to the active neutrino mass differences $m_{atm}$ and $m_{sol}$. Recalling that $M_D$ is defined as the product of the Yukawa matrix $F$ and the Higgs vev one finds for the two flavours case

$$M_\nu = v^2 \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} 1/M_1 & 0 \\ 0 & 1/M_2 \end{pmatrix} \begin{pmatrix} F_{11} & F_{21} \\ F_{12} & F_{22} \end{pmatrix}$$

because the Majorana masses are assumed to be almost degenerate. The sign of the mass matrix is unconventional which is cured by the following procedure. From the diagonalization one has

$$\begin{pmatrix} \nu_L, N^c \end{pmatrix} \begin{pmatrix} 0 & M_D \\ M_D^T & M \end{pmatrix} \begin{pmatrix} \nu_L^c \\ N \end{pmatrix} = \begin{pmatrix} \nu_L, N^c \end{pmatrix} U^T \begin{pmatrix} M_\nu & 0 \\ 0 & M \end{pmatrix} K U \begin{pmatrix} \nu_L^c \\ N \end{pmatrix}$$

with $K = \text{diag}(-1, 1)$. If we now define the following fields

$$\begin{pmatrix} n_{1L} \\ n_{2L} \end{pmatrix} = U \begin{pmatrix} \nu_L \\ N^c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n_{1R} \\ n_{2R} \end{pmatrix} = K U \begin{pmatrix} \nu_L^c \\ N \end{pmatrix}$$

we have that the mass term has correct signs and that $n_i = n_{iR} + n_{iL}$ satisfy the Majorana condition written in Appendix A with

$$n_1^c = -n_1 \quad \text{and} \quad n_2^c = n_2,$$

and the mass term has conventional sign. Now if, for example, we choose couplings such that

$$F_{11}, F_{22} \gg F_{12}, F_{21},$$

the $M_D M^{-1} M_D^T$ takes almost a diagonal form and one can write

$$\begin{pmatrix} m_{atm} & 0 \\ 0 & m_{sol} \end{pmatrix} \approx \frac{v^2}{M} \begin{pmatrix} F_{11}^2 & 0 \\ 0 & F_{22}^2 \end{pmatrix}$$

\footnote{In order to avoid confusion, $\theta$ is a matrix, and we mean that all the elements of this matrix are small.}
such that
\[ F_{11} = \sqrt{\frac{M_{atm}}{v^2}} \quad \text{and} \quad F_{22} = \sqrt{\frac{M_{sol}}{v^2}} \] (1.14)

where \( m_{atm} \) and \( m_{sol} \) are given by

\[ m_{atm} = \sqrt{\Delta m^2_{atm}} \simeq 50 \text{ meV} \quad \text{and} \quad m_{sol} = \sqrt{\Delta m^2_{sol}} \simeq 9 \text{ meV} \] (1.15)

and respectively measured by [A+04] and [A+05]. After being done with finding a set of Yukawa couplings giving correct light neutrinos masses we can write an interaction term between light neutrinos, Majorana neutrinos and the Higgs field with some restrictions on the \( F \) matrix elements. We will also define the hermitian matrix \( K \) as

\[ K = F^\dagger F \quad \text{and} \quad K_{kl} = \sum_i (F_i^\dagger)_{ki} F_{il} \] (1.16)

The matrix \( K \) will be useful as we advance further.

We will restrict the discussion to the case of two RH flavors only. Generalization to three flavors is straightforward. Moreover, this can be also justified when one of the RH neutrinos is very weakly coupled which yields to the zero mass of one of the active neutrinos. In that case the weakly coupled RH neutrino does not participate in leptogenesis, while \( m_{atm} \) and \( m_{sol} \) become physical masses of the rest two active neutrinos.
2. FINITE TEMPERATURE FIELD THEORY

In this section we are going to discuss two main formalisms which have been developed to describe finite temperature effects in quantum field theory. There are three different approaches to cope with finite temperature effects, namely Matsubara formalism, real time formalism or closed path integral formalism and thermo field dynamics. The latter will not be discussed here since the first two are sufficient for our purposes. One can be referred to the third chapter of [Das97] for a detailed discussion of the finite temperature field theory (FTFT). Below we will briefly remind some technicalities.

2.1 Matsubara Formalism

The most crucial observation lies in the fact that the density matrix has the form of a time evolution operator for negative imaginary times. Indeed a system density matrix is defined by

$$\rho(\beta) = e^{\beta H} = e^{i(-i\beta)H}$$ (2.1)

where $\beta$ is the inverse temperature. We will use a variable $\tau$ which belongs to the interval $0 \leq \tau \leq \beta$. For a system described by the Hamiltonian $H = H_0 + H_{\text{int}}$ the density matrix can be written in the form

$$\rho(\beta) = e^{\beta H} = e^{\beta H_0}e^{-\beta H_0}e^{\beta H} = \rho_0(\beta)S(\beta)$$ (2.2)

The equation of motion for $S = \rho_0^{-1}\rho$ can now be derived and one obtains

$$S(\beta) = T_\tau \left(e^{-\int_0^\beta d\tau H_\text{int}^I}\right).$$ (2.3)

where the superscript $I$ is for the Hamiltonian in the interaction picture. From here everything is similar to the zero-temperature field theory. In particular Wicks theorem has its finite temperature formulation and $S$ satisfies the semigroup properties. We now can define the two-point Green function as

$$G_\beta = \langle T_\tau(\phi_H(\tau)\phi_H^I(\tau'))\rangle_\beta.$$ (2.4)

where the ensemble average is defined as $\langle A \rangle_\beta = Z^{-1}\text{Tr}(\rho A)$ with $Z = \text{Tr}\rho$. The essential point is that unlike the usual zero-temperature Green function the interval where $\tau$ belongs is finite. Therefore, the Fourier transformation only
involves discrete frequencies which are different for bosons and fermions. This comes from the fact that under the $\tau$-ordering one has to commute fields and fermionic expressions pick up a minus sign when commuted. We thus have

$$G(\vec{x}, \tau) = \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} e^{-i(\omega_n \tau - \vec{k}\vec{x})} G(\vec{k}, \omega_n).$$

(2.5)

Propagators can now be computed for any theory. Bosonic and fermionic equations for the Green functions are respectively

$$(\Box + m^2)G_B(x) = -i\delta^4(x)$$

$$(i\not\partial - m)G_F(x) = i\delta^4(x),$$

(2.6)

which lead to the following Fourier components

$$G_B(\vec{k}, \omega_n) = \frac{1}{\omega_n^2 - \vec{k}^2 - m^2}$$

$$G_F(\vec{k}, \omega_n) = \frac{\gamma_0 \omega_n - \gamma\vec{k} + m}{\omega_n^2 - \vec{k}^2 - m^2}$$

(2.7)

where the frequencies are given by

$$\omega_n = \begin{cases} 
\frac{2n\pi i}{\beta} & \text{for bosons} \\
\frac{(2n+1)\pi i}{\beta} & \text{for fermions}
\end{cases}$$

(2.8)

It is now clear that the Feynman rules are the same as those of zero temperature field theory where we replace the usual propagators with the Matsubara ones and the integration over the energy with the sum over frequencies

$$\int \frac{d^4k}{(2\pi)^4} \rightarrow \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3}$$

(2.9)

where $n \in \mathbb{Z}$. The computations in Matsubara involve infinite sums. The most elegant method to compute them is to use the residue theorem. Let us introduce a generic function $R_X(\omega_n)$ with $X \in \{B, F\}$ which indicates if the sum is to perform over fermionic or bosonic frequencies. Let us suppose we want to compute the following sum

$$\frac{1}{\beta} \sum_n R_X(\omega_n)$$

(2.10)

If $R_X$ does not have any poles along the imaginary axis we can multiply it by a function $Q_X$ which have poles of order one and residue equal to one and integrate the product around the imaginary axis. The function $Q_X$ is, of course, different whether one considers fermionic or bosonic frequencies. We can rewrite the sum (2.10) as the following integral

$$\frac{1}{2\pi i} \oint_C dz R_X(z)Q_X(z)$$

(2.11)
where the $Q_X(z)$ functions are given by

$$
\begin{cases}
Q_B(z) = \frac{1}{2} \coth \left( \frac{z\beta}{2} \right) \\
Q_F(z) = \frac{1}{2} \tanh \left( \frac{z\beta}{2} \right)
\end{cases}
$$

(2.12)

The fact the $\tanh$ and $\coth$ functions which appear here can be easily related to $n_F$ and $n_B$ through

$$\coth \left( \frac{E\beta}{2} \right) = 1 + 2n_B(E) \quad \text{and} \quad \tanh \left( \frac{E\beta}{2} \right) = 1 - 2n_F(E)$$

(2.13)

The $C$ contour can be deformed into the one shown on figure 2.1 if $R_X Q_X$ sufficiently fast go to zero when $|z|$ goes to infinity. Because $Q_X$ have poles on the imaginary axis the only poles inside the two contours are those of $R_X$. Thus we have

$$\frac{1}{\beta} \sum_n R_X(\omega_n) = - \sum \text{Res} \left[ R_X(z)Q_X(z) \right]$$

(2.14)

where the minus sign comes from the fact that we changed the contour orientation.

### 2.2 Closed Path Formalism

The second formalism we introduce is also based on a system with the density matrix defined in a quantum theoretical way as

$$\rho = \sum_n p_n \vert \psi_n(t) \rangle \langle \psi_n(t) \vert$$

(2.15)
If the $p_n$ is constant the density can be written as

$$\rho = U(t, 0) \rho(0) U(0, t)$$

(2.16)

This case corresponds to the adiabatic system since the entropy is defined as $- \sum p_n \log p_n$. The $U$ operator satisfies

$$U(t, t') = T \left( e^{-i \int_{t'}^{t} dt'' H(t'')} \right)$$

(2.17)

Up to this step everything was classical. Let us now define the Hamiltonian of the system as being $H_I$ for $t < 0$ and $H(t)$ for $t > 0$. We should even be more precise and distinguish between $\text{Re}(t) \leq 0$. Then we can assume that there exists a time $T$ sufficiently negative so that we can write

$$\rho(0) = \frac{U(T - i\beta, T)}{\text{Tr} U(T - i\beta, T)}$$

(2.18)

in such a way that the Hamiltonian in (2.17) is $H_I$. This expression for $\rho(0)$ satisfies all properties it has to. It is a positive hermitian operator with unit trace. We now have a new expression for $\rho$ which include $\beta$ in its $\rho(0)$ part. We can now compute the expectation value of an observable $A$ as $\langle A \rangle = \text{Tr} \rho A$.

Now we can draw a contour in the $t$-complex plane as shown on figure 2.2. In [LvW87] the authors generalize this contour by shifting the $C_-$ part in the lower part of the complex plane such that $\text{Im}(C_-) = -\sigma$. All these contours lead to an equivalence class of finite temperature theories. From the form of the expectation value $\langle A \rangle$ one can deduce a generating functional from which we can give a path integral representation

$$Z[J_c] = \int \mathcal{D}\phi e^{i \int_c dt \int d^3x (L + J_c \phi)}$$

(2.19)

The Green functions are then obtained by computing the second derivative of the generating functional. One can show that the $C_3$ part of the contour gives no contribution and one can close the contour. Then the Green function $G(t - t')$ has to be computed in the case when both $t$ and $t'$ are on $C_+$, both on

![Fig. 2.2: Contour in the t-complex plane](image-url)
2. Finite Temperature Field Theory

\[ C_-, t \text{ on } C_+ \text{ and } t' \text{ on } C_- \text{ and } t \text{ on } C_- \text{ and } t' \text{ on } C_+ \]. This leads to the following definitions

\[
\begin{align*}
  iG_{++}(t - t') &= \langle T(\phi(t)\phi(t')) \rangle \\
  iG_{+-}(t - t') &= \langle \phi(t')\phi(t) \rangle \\
  iG_{-+}(t - t') &= \langle \phi(t)\phi(t') \rangle \\
  iG_{--}(t - t') &= \langle T^{-1}(\phi(t)\phi(t')) \rangle
\end{align*}
\]  

(2.20)

One can refer to [LvW87] or [Das97] for the complete derivation of the Green functions. Here we will give their representation in momentum space. First the bosonic Green functions

\[
\begin{pmatrix}
  D_{++} & D_{+-} \\
  D_{-+} & D_{--}
\end{pmatrix}
= 
\begin{pmatrix}
  D & 0 \\
  0 & -iD^*
\end{pmatrix}
- 2\pi i
\begin{pmatrix}
  n_B(p) & n_B(p) + \theta(-p^0) \\
  n_B(p) & n_B(p)
\end{pmatrix}
\delta(p^2 - m^2)
\]  

(2.21)

Then the fermionic ones

\[
\begin{pmatrix}
  S_{++} & S_{+-} \\
  S_{-+} & S_{--}
\end{pmatrix}
= (\slashed{p} + m)
\begin{pmatrix}
  D & 0 \\
  0 & -iD^*
\end{pmatrix}
+ 2\pi i
\begin{pmatrix}
  n_F(p) & n_F(p) - \theta(p^0) \\
  n_F(p) & n_F(p)
\end{pmatrix}
\delta(p^2 - m^2)
\]  

(2.22)

where we have defined \( D \) as the usual zero-temperature bosonic Green function and \( n_B \) and \( n_F \) as the boson and fermion statistics

\[
D = \frac{1}{p^2 - m^2 + i\epsilon} \quad \text{and} \quad n_{B/F}(p) = \frac{1}{e^{\beta|p^0|} + 1}
\]  

(2.23)

As it was previously mentioned we can generalize this expression by shifting the lower part of the contour in the imaginary part of the complex plane. Then the Green functions are slightly modified. For \( \sigma = \beta/2 \) one can show the theta functions in the temperature dependent part of the Green functions vanish and the statistics are multiplied by \( e^{\beta|p^0|/2} \). The fact that propagators acquire a two dimensional matrix structure is a general feature of finite temperature theories. This is often referred as the doubling of degrees of freedom. Advanced and retarded Green functions can simply be found as \( G_R = G_{++} - G_{+-}, \ G_A = G_{++} - G_{-+} \) and \( G_C = G_{++} + G_{--} \) from their definitions in terms of fields commutator. \( G_C \) is often referred as the correlated Green function. Furthermore one can show that \( \text{Re}G_{++} = \text{Re}G_R \) which is a very nice property that we will use when computing our system’s Hamiltonian.

We have now seen the two most important way to deal with finite temperature Green functions. Both have their advantages and disadvantages. Matsubara formalism is only valid for close to the equilibrium sets of particles while
closed contour formalism has some ill-defined product of $\delta$-functions when multiplying two physical propagators and one has to use the propagators in matrix form (2.21) and (2.22) instead of physical ones.
3. QUANTUM KINETIC EQUATION

3.1 Derivation of the QKE

The Quantum Kinetic Equation (QKE) can be read from [SR93]

\[ i \dot{\rho} = [\rho, \mathcal{H}] - \frac{i}{2} \{ \Gamma^d, \rho \} + \frac{i}{2} \{ \Gamma^p, 1 - \rho \} \tag{3.1} \]

which is valid for all modes. \( \mathcal{H} \) is the Hamiltonian of the system and \( \Gamma^d \) and \( \Gamma^p \) are respectively the destruction and production rates of the Majorana neutrino. The density matrix \( \rho \) is the system’s density matrix. This introduces mixing between Leptons and Majorana neutrinos. Usually the \( \rho \) matrix is written as

\[
\rho = \begin{pmatrix}
\rho_{LL} & \rho_{LT} & \rho_{LN} & \rho_{LN} \\
\rho_{TL} & \rho_{TT} & \rho_{TN} & \rho_{TN} \\
\rho_{NL} & \rho_{NT} & \rho_{NN} & \rho_{NN} \\
\rho_{NL} & \rho_{NT} & \rho_{NN} & \rho_{NN}
\end{pmatrix}
\tag{3.2}
\]

Removing the \( \Gamma \)’s from the QKE leads to the well-known equation of the evolution of an operator in the Heisenberg picture. One can also rewrite QKE in the following form:

\[ i \dot{\rho} = [\rho, \mathcal{H}] - \frac{i}{2} \{ \Gamma^d + \Gamma^p, \rho \} + i \Gamma^p \tag{3.3} \]

Let us now modify the QKE by taking in account that the equilibrium density matrix \( \rho^{eq} \) satisfies

\[ \{ \Gamma^d_k, \rho^{eq}_k \} = \{ \Gamma^p_k, 1 - \rho^{eq}_k \} \tag{3.4} \]

Then QKE can be rewritten as

\[ i \dot{\rho}_k = [\rho_k, \mathcal{H}_k] - \frac{i}{2} \{ \Gamma^d_k + \Gamma^p_k, \rho_k - \rho^{eq}_k \} \tag{3.5} \]

We now want to transform it as a differential equation over the with respect to the time like variable \( z \propto \sqrt{t} \). First let us notice that

\[ \frac{d}{dt} = \frac{dT}{d\tau} \frac{d\tau}{dz} = \frac{dT}{dT} \frac{dz}{d\tau} \tag{3.6} \]

From Friedmann equation [KT90] one finds that for a flat space during the radiation dominated epoch the Hubble parameter and the temperature are related through

\[ H^2 = \frac{8\pi G}{3} \rho_{\text{rad}} = \frac{8\pi}{3M_{\text{Pl}}^2} \frac{\pi^2}{30} g_\ast T^4 \rightarrow T = \sqrt{M_0 H} \tag{3.7} \]
where

\[ M_0 = M_{Pl} \sqrt{\frac{45}{4\pi^3 g_\ast}} \]  

(3.8)

where \( M_{Pl} \) is the Planck mass and \( g_\ast \) the number of degrees of freedom. Thus

\[ \frac{dT}{dt} = \frac{1}{2} \left( \frac{M_0}{H} \right)^{1/2} \frac{dH}{dt} \]  

(3.9)

As we consider the radiation dominated epoch the Hubble parameter evolution is given by

\[ H = \frac{1}{2t} \rightarrow \frac{dH}{dt} = -\frac{1}{2t^2} = -2H^2 \]  

(3.10)

which leads to

\[ \frac{dT}{dt} = -M_0^{1/2}H^{3/2} = -\frac{1}{M_0}T^3 \]  

(3.11)

Moreover from the definition of \( z \) one has

\[ z = \frac{M}{T} \rightarrow \frac{dz}{dT} = -\frac{M}{T^2} = -\frac{z}{T} \]  

(3.12)

Then (3.6) can be written as

\[ \frac{d}{dt} = \left( \frac{T^3}{M_0} \right) \left( \frac{z}{T} \right) \frac{d}{dz} = \frac{M^2}{zM_0} \frac{d}{dz} \]  

(3.13)

All this, finally, leads to the following form of the Quantum Kinetic Equation

\[ \frac{iM^2}{zM_0} \frac{d}{dz} = [\rho_k, \mathcal{H}_k] - \frac{i}{2} \{ \Gamma^d_k + \Gamma^p_k, \rho_k - \rho_k^{eq} \} \]  

(3.14)

The decay and the production rates will be computed in chapter 5. We now proceed with the computation of the Hamiltonian for a given mode \( k \).
4. HAMILTONIAN

4.1 Definition of the Hamiltonian

To compute the Hamiltonian we will first consider a one-loop renormalized theory. The propagator is then modified by the self-energy diagram. The self-energy can be written as the sum of two contributions

$$\Sigma = \Sigma(T = 0) + \Sigma'$$  \hspace{1cm} (4.1)

where $\Sigma'$ is the finite-T corrections. Assuming the theory has already been renormalized at zero-temperature the fermion propagator can be written as

$$\frac{1}{K - M - \Sigma'},$$  \hspace{1cm} (4.2)

where $M$ is the fermion’s zero-temperature physical mass. Modifying the propagator implies changes at the Lagrangian level. The kinetic and mass terms can be written as

$$\bar{N}_i O_{ij} N_j$$  \hspace{1cm} (4.3)

where $N$ is the fermion’s wave function. Renormalization implies that the $O$ operator takes the following form

$$O_{ij} = \delta_{ij}(i\partial - M_i) + \Sigma'_{ij}$$  \hspace{1cm} (4.4)

From this Lagrangian we can now compute the Hamiltonian. First of all we must write the kinetic term in a canonical way and then compute the Hamiltonian as usually:

$$H = \sum_i \frac{\partial \mathcal{L}}{\partial (i\partial N_i)} \partial_0 N_i - \mathcal{L},$$  \hspace{1cm} (4.5)

which can be written in the matrix form as $\bar{N}_i H_{ij} N_j$. We are now going to compute the finite temperature contribution to the self-energy in order to compute the Hamiltonian of our system.

4.2 Self-energy computation

As mentioned by [Wel82] Im$\Sigma'$ is small compared to Re$\Sigma'$ and thus will be dropped in our computation. Then for a particle with 4-momentum $K^\mu$ the
thermal contribution to the self-energy at one loop has the following form
\[ \text{Re} \Sigma' = -a K - b \psi - c M \] (4.6)

The \(a\), \(b\) and \(c\) functions have to be Lorentz-invariant and can depend on the following scalars
\[ \begin{cases} \omega = K^\mu u_\mu \\ k = \sqrt{(K^\mu u_\mu)^2 - K^2} \end{cases} \] (4.7)

An interesting property of these definitions is that \(K^2 = \omega^2 - k^2\) and thus \(\omega\) can be interpreted as the energy of the particle and \(k\) as the modulus of its momentum. In many computations we will choose the rest frame of the heat bath where \(u^\mu = (1, \vec{0})\) and thus we will have \(\omega = K^0\). From (4.6) we have that
\[ a = \frac{1}{4} \left[ \frac{1}{k^2} \text{Tr}(K \text{Re}\Sigma') - \omega \frac{1}{4} \text{Tr}(\psi \text{Re}\Sigma') \right] \] (4.8)

\[ b = \frac{1}{4} \left[ (\omega^2 - k^2) \frac{1}{4} \text{Tr}(\psi \text{Re}\Sigma') - \omega \frac{1}{4} \text{Tr}(K \text{Re}\Sigma') \right] \] (4.9)

\[ c = -\frac{1}{M} \frac{1}{4} \text{Tr}(\text{Re}\Sigma') \] (4.10)

We can now move on the computation of the self-energy diagram.

**Self-energy diagram**

In this computation we will use the real time formalism in which usual rules of zero-temperature QFT do apply if one uses thermal substitutes for the zero-temperature propagators. Thermal propagators are given by (2.22) for fermions and (2.21) for scalar bosons. The expression for gauge bosons in the Feynman gauge is given by
\[ -\eta_{\mu \nu} D(p). \]

As previously mentionned \(\text{Re} G_R = \text{Re} G_{++}\) so that the only propagators needed are
\[ S_{++}(p) = (\not{p} + M_1) \left( \frac{1}{p^2 - M_1^2 + i\epsilon} + i\Gamma_F(p) \right) \] (4.11)

\[ D_{++}(p) = \frac{1}{p^2 - M_1^2 + i\epsilon} - i\Gamma_B(p) \] (4.12)

where \(\Gamma_{F/B} = 2\pi \delta(p^2 - M_1^2)n_{F/B}(|p \cdot u|)\) and \(u^\mu\) the 4-velocity of the thermal bath. We are now able to compute the self-energy diagram shown on Fig. 4.1.

\[ \Sigma = iK \int \frac{d^4p}{(2\pi)^4} S_{++}(p + K)D_{++}(p) \] (4.13)

One should note that \(\Sigma\) is a matrix in the flavor space. The presence of the \(K\) matrix in the previous expression is due to the fact that one has to sum over the different leptonic intermediate states. One can now straightforwardly write
Fig. 4.1: Yukawa contribution to the Majorana self-energy

\[ \text{Re}\Sigma' = K \int \frac{d^4p}{(2\pi)^4} (\bar{p} + \bar{K} + M_1) \left[ \frac{\Gamma_B(p)}{(p + K)^2 - M_1^2} - \frac{\Gamma_F(p + K)}{p^2 - M_2^2} \right] \] (4.14)

Then the traces we need to compute in order to get the functions \( a, b \) and \( c \) given in equations (4.8) to (4.10) are

\[ \frac{1}{4} \text{Tr}(\text{Re}\Sigma') = K M_1 \int \frac{d^4p}{(2\pi)^4} \left[ \frac{\Gamma_B(p)}{(p + K)^2 - M_1^2} - \frac{\Gamma_F(p + K)}{p^2 - M_2^2} \right] \] (4.15)

\[ \frac{1}{4} \text{Tr}(\bar{K}\text{Re}\Sigma') = K \int \frac{d^4p}{(2\pi)^4} (K^2 + p\cdot K) \left[ \frac{\Gamma_B(p)}{(p + K)^2 - M_1^2} - \frac{\Gamma_F(p + K)}{p^2 - M_2^2} \right] \] (4.16)

\[ \frac{1}{4} \text{Tr}(\bar{u}\text{Re}\Sigma') = K \int \frac{d^4p}{(2\pi)^4} (u\cdot p + u\cdot K) \left[ \frac{\Gamma_B(p)}{(p + K)^2 - M_1^2} - \frac{\Gamma_F(p + K)}{p^2 - M_2^2} \right] \] (4.17)

A detailed computation of these integrals is presented in the Annex B without taking into account any assumptions about lepton, Higgs and Majorana masses. In our case we can set \( c \) to zero because it is proportional to the mass of the lepton doublet \( L \) which is zero before electroweak phase transition as one can notice from (4.10) and (4.15). The integrals over the 4-momentum can be reduced to an integral over the norm of the 3-momentum by integrating over the energy and using the \( \delta \)-distribution appearing in \( \Gamma_B \) and \( \Gamma_F \). The integrals over the solid angle are also easily performed. Applying this procedure we get the following expressions for (4.16) and (4.17) where we have set the \( L \) mass and the Higgs mass to zero to simplify our computations. Of course, putting the Higgs mass to zero is not correct and by doing such an approximation one neglects CP-violating decays of the Higgs field. Computing the QKE with massive Higgs would be the natural continuation of this work. With the following definitions

\[ L_{1, \pm} = \log \left( \frac{\omega^2 - k^2 - 2p\omega - 2kp}{\omega^2 - k^2 + 2p\omega + 2kp} \right) \pm \log \left( \frac{\omega^2 - k^2 + 2p\omega - 2kp}{\omega^2 - k^2 + 2p\omega + 2kp} \right) \] (4.18)
\[ L_2^\pm = \log \left( \frac{\omega^2 - k^2 + 2p\omega + 2kp}{\omega^2 - k^2 + 2p\omega - 2kp} \right) \pm \log \left( \frac{\omega^2 - k^2 - 2p\omega + 2kp}{\omega^2 - k^2 - 2p\omega - 2kp} \right) \] (4.19)

one obtains

\[ \frac{1}{4} \text{Tr}(\mathcal{K} \text{Re} \Sigma') = K \int \frac{dp}{16\pi^2k} \left[ n_B(\epsilon_2) L_2^+ + n_F(\epsilon_1) L_1^+ \right] \] (4.20)

\[ \frac{1}{4} \text{Tr}(\mathcal{F} \text{Re} \Sigma') = K \int \frac{dp}{16\pi^2k} \left[ n_B(\epsilon_2) \left( L_2^- + \frac{\omega}{\epsilon_2} L_2^+ \right) + n_F(\epsilon_1) L_1^- \right] \] (4.21)

where

\[ L_i^+ = 4pk + \frac{\omega^2 - k^2}{2} L_i^+ \] (4.22)

As previously explained we do not need to compute (4.15) here. The result is nevertheless given in Annex B. As we can notice \( \omega \) enters the previous equations. We now would like to express it as a functions of \( k \) and \( M \). In other words we would like to know how the dispersion relation has been modified by the one-loop renormalization we have performed. First we are going to compute the modified propagator and then find its poles.

**Modified Propagator**

From the result of the previous integrals one can compute the values of (4.8), (4.9) and (4.10) which are related to the self-energy by (4.6). The fermion propagator can thus be written as

\[ S = \frac{1}{\mathcal{K} - M + a\mathcal{K} + b\mathcal{F} + cM} \] (4.23)

where we again neglected the imaginary part of \( \Sigma' \). The most general forms we can get for \( a, b \) and \( c \) are their decomposition in the chiral basis. One can write

\[ X = X_LP_L + X_RP_R \quad \text{where} \quad X \in \{a, b, c\} \] (4.24)

where \( P_L \) and \( P_R \) are the usual projectors on the left and right chiralities respectively. Following [Wel82] we define

\[ X^\mu = (1 + a_X)K^\mu + b_X u^\mu \quad \text{where} \quad X \in \{L, R\} \] (4.25)

and \( M_R \) and \( M_L \) as respectively \( M(1 - c_L) \) and \( M(1 - c_R) \). The aim is now to find the inverse operator of the denominator of our propagator. If our theory has a chiral symmetry one has \( M = M_R = M_L = 0 \) since mass terms break chiral symmetry. The inverse propagator would then be

\[ S^{-1} = LP_L + RP_R \] (4.26)

As \( \gamma^5 \) anticommutes will all \( \gamma^\mu \) matrices and \( P_L^2 = P_L, P_R^2 = P_R \) and \( P_LP_R = 0 \) one can rewrite this term as

\[ S^{-1} = P_RLP_L + P_LRP_R \] (4.27)
One can now inverse the previous operator without having to do any laborious decomposition over the 16 matrices generating $M_4(\mathbb{C})$. This can be done by using the orthogonality of the projectors and the fact that $P_L + P_R = 1$. It is straightforward to see that the inversion of (4.27) is given by

$$S = P_L \frac{L}{L^2} P_R + P_R \frac{R}{R^2} P_L$$

(4.28)

In our case the operator we have to inverse is slightly more complicated (4.23) but the technique we have used is the same. The numerator of the propagator is then given by the following expression

$$(L^2 \tilde{R} - M_L M_R \tilde{L}) P_L + (R^2 \tilde{L} - M_L M_R \tilde{R}) P_R$$

$$+ M_R P_L (L \cdot R - M_L M_R + \frac{1}{2} [L, \tilde{R}]) P_L$$

$$+ M_L P_R (L \cdot R - M_L M_R + \frac{1}{2} [\tilde{L}, R]) P_R$$

(4.29)

which reproduces the expression of [Wel82] and corrects the [QVC95] one. The denominator can now be found by multiplying the previous expression by the denominator of (4.23). It is given by

$$L^2 R^2 - 2 M_L M_R (L \cdot R) + M_L^2 M_R^2$$

(4.30)

One can check that this results nicely reproduces the chirally symmetric example we have previously derived.

**Dispersion Relation**

We have found the one loop renormalized fermion propagator. We can now compute the dispersion relation which is given by the position of the poles of the renormalized propagator. In the case we are interested in $X = X_R = X_L$ for $X \in \{a, b, M\}$ and $c$ is zero so that the denominator reduces to

$$L^4 - 2 M^2 L^2 + M^4 = 0 \rightarrow L^2 = M^2$$

(4.31)

Replacing $L$ by its definition yields to

$$(1 + a)^2 K^2 + b^2 + 2 b (1 + a) K \cdot u = M^2$$

$$(1 + a)^2 (\omega^2 - k^2) + b^2 + 2 b (1 + a)\omega = M^2$$

$$(1 + a)^2 (\omega^2 + b^2) = M^2 + (1 + a)^2 k^2$$

(4.32)

where we used the rest frame of the heat bath. The positive energy branch is then given by

$$\omega = \sqrt{\frac{M^2}{(1 + a)^2} + k^2 - \frac{b}{1 + a}}$$

(4.33)
which can safely be approximated by $\sqrt{k^2 + M^2}$ if both $a$ and $b$ are small which is the case here because both $a$ and $b$ are proportional to $K$ which contains squared elements of the Yukawa matrix $F$. We can now simplify the expressions (4.8) to (4.10). We also introduce the following rescaling

$$k \rightarrow kT \quad p \rightarrow pT \quad M \rightarrow zT \quad \omega \rightarrow \omega T$$

(4.34)

so that $k$, $p$ and $\omega$ are constants with respect to time. Then $L^\pm_1$ and $L^\pm_2$ become

$$L^+_1 = \log \left( \frac{z^2 - 2p\sqrt{k^2 + z^2} - 2kp}{z^2 - 2p\sqrt{k^2 + z^2} + 2kp} \right) \pm \log \left( \frac{z^2 + 2p\sqrt{k^2 + z^2} - 2kp}{z^2 + 2p\sqrt{k^2 + z^2} + 2kp} \right)$$

(4.35)

$$L^+_2 = \log \left( \frac{z^2 + 2p\sqrt{k^2 + z^2} + 2kp}{z^2 + 2p\sqrt{k^2 + z^2} - 2kp} \right) \pm \log \left( \frac{z^2 - 2p\sqrt{k^2 + z^2} + 2kp}{z^2 - 2p\sqrt{k^2 + z^2} - 2kp} \right)$$

(4.36)

and (4.16) and (4.17) take the following form

$$\frac{1}{4} \text{Tr}(K \text{Re} \Sigma') = K \int_0^\infty \frac{dp}{16\pi^2 z^2} M^2 \left[ n_B(p) \left( 4p + \frac{z^2 L^+_2}{2k} \right) + n_F(p) \left( 4p + \frac{z^2 L^+_1}{2k} \right) \right]$$

(4.37)

$$\frac{1}{4} \text{Tr}(\gamma \text{Re} \Sigma') = K \int_0^\infty \frac{dp}{16\pi^2 z k} \left[ n_B(p) \left( L^-_2 + \frac{\sqrt{k^2 + z^2}}{p} L^+_2 \right) + n_F(p) L^+_1 \right]$$

(4.38)

where one has to be careful that according to the performed rescaling the statistics factors are modified in the following manner

$$n_B(p) = \frac{1}{e^p - 1} \quad \text{and} \quad n_F(p) = \frac{1}{e^p + 1}$$

(4.39)

These integrals are going to be performed numerically but one can compute their high temperature asymptotics. The behaviour of $L^\pm_1$ for large momentum $p$ can be easily estimated. First the integrals over $p n_X(p)$ with $X \in \{B, F\}$ are given by

$$\int_0^\infty dp n_B(p) p = \frac{\pi^2}{6} \quad \text{and} \quad \int_0^\infty dp n_F(p) p = \frac{\pi^2}{12}$$

(4.40)

Then one can notice that in our approximation where both $M_1$ and $M_2$ have been set to zero the $L^\pm_1$ expressions are related through

$$L^+_1(p) = -L^-_2(p) \quad \text{and} \quad L^-_1(p) = L^+_2(p)$$

(4.41)

Let us define $L_1(p)$ and $L_2(p)$ as

$$\begin{cases}
L_1(p) & \equiv L^+_1(p) \\
L_2(p) & \equiv 2 \log \left( \frac{\omega_+}{\omega_-} \right) - L^-_2(p)
\end{cases}$$

(4.42)
with \( \omega_\pm \equiv \frac{1}{2}(\omega \pm k) \) so that our notations rejoin [Wel82]. Then the \( L_1 \) and \( L_2 \) functions can be rewritten

\[
L_1(p) = \log \left( \frac{p + \omega_+}{p + \omega_-} \right) - \log \left( \frac{p - \omega_+}{p - \omega_-} \right)
\]

\[
L_2(p) = \log \left( \frac{p + \omega_+}{p + \omega_-} \right) + \log \left( \frac{p - \omega_+}{p - \omega_-} \right)
\]

whose behaviour for \( p \to \infty \) are given by

\[
\lim_{p \to \infty} L_1(p) = \lim_{p \to \infty} \left[ \log \left( 1 + \frac{\omega_+}{p} \right) - \log \left( 1 - \frac{\omega_+}{p} \right) \right] = \frac{2}{p}\left( \omega_+ - \omega_- \right) = \frac{2k}{p}
\]

and

\[
\lim_{p \to \infty} L_2(p) = \lim_{p \to \infty} \left[ \log \left( 1 + \frac{\omega_+}{p} \right) + \log \left( 1 - \frac{\omega_+}{p} \right) \right] = \log \left( 1 + \frac{\omega_+}{p} \right) + \log \left( 1 - \frac{\omega_+}{p} \right)
\]

\[
= -\left( \frac{\omega_+}{p} \right)^2 + \left( \frac{\omega_-}{p} \right)^2 = -\frac{k(2k^2 + z^2)}{p^2}
\]

where we used the approximative dispersion relation in the last step. The dominant term for large \( T \) of (4.37) is then

\[
\frac{1}{4} \text{Tr}(\mathcal{K} \Re \Sigma') \simeq K \int_0^\infty \frac{dp}{16\pi^2} \frac{M^2}{x^2} \left( n_B(p) + n_F(p) \right) 4p = K \frac{M^2}{16\pi^2} \]

because the other terms are proportional to \( 1/p \) times \( n_X \) with \( X \in \{B, F\} \). The integral over these terms can be roughly approximated by the integral from zero to \( O(1) \) since \( n_X \) acts like a cutoff. The same can be done for (4.38). The integrals over \( L_i \) can again be safely neglected and we stay with the following
leading expression for high temperatures

\[
\frac{1}{4} \text{Tr}(\text{Re} \Sigma') \simeq K \int_0^\infty \frac{dp}{16\pi^2} \frac{M_p}{z k} \left( n_B(p) + n_F(p) \right) 2 \log \left( \frac{\omega_+}{\omega_-} \right)
\]

\[
= K \frac{M}{32z k} \log \left( \frac{\sqrt{k^2 + z^2 + k}}{\sqrt{k^2 + z^2 - k}} \right)
\]

(4.47)

Inserting (4.46) and (4.47) in (4.8) and (4.9) gives the following asymptotics for the \(a\) and \(b\) factors appearing in (4.6)

\[
a \simeq \frac{K}{16k^2} \left[ 1 - \frac{\sqrt{k^2 + z^2}}{2k} \log \left( \frac{\sqrt{k^2 + z^2 + k}}{\sqrt{k^2 + z^2 - k}} \right) \right]
\]

(4.48)

\[
b \simeq \frac{KM}{16kz} \left[ \frac{z^2}{2k^2} \log \left( \frac{\sqrt{k^2 + z^2 + k}}{\sqrt{k^2 + z^2 - k}} \right) - \frac{\sqrt{k^2 + z^2}}{k} \right]
\]

(4.49)

We now have all expressions we need to compute the real part of the finite temperature contribution to the self-energy. Equations (4.37) and (4.38) are in the exact form which can be used if doing numerics. High temperature asymptotics are given by (4.46) and (4.47). Using these results we are now able to compute the Hamiltonian according to the procedure described in the beginning of the chapter.

### 4.3 Hamiltonian Computation

#### 4.3.1 Kinetic Term

The matrix form of the kinetic term of the renormalized Lagrangian has the following form

\[
A_{\text{kin}} \equiv \begin{pmatrix}
1 - a_{11} & -a_{12} \\
-a_{21} & 1 - a_{22}
\end{pmatrix}
\]

(4.50)

where \(a\) which is computed in Annex B is real and \(K\) is hermitian. Its matrix structure comes from the \(K\) matrix contained in its expression. We now have to diagonalize \(1 - aK\) which is also hermitian. We would moreover like to diagonalize it by a unitary matrix. This can be done if \(A_{\text{kin}}\) is normal, i.e. if \(A_{\text{kin}}^{\dagger}A_{\text{kin}} = A_{\text{kin}}A_{\text{kin}}^{\dagger}\) which is obviously the case here \(A_{\text{kin}}\) being hermitian. As the hypothesis on the normality of \(A_{\text{kin}}\) is over-satisfied we can have more information on the form of the \(U\) matrix which will diagonalize it. Let \(D\) be an hermitian matrix. Then it can be written as

\[
D = \begin{pmatrix}
A & B e^{i\delta} \\
B e^{-i\delta} & C
\end{pmatrix}
\]

(4.51)

with \(A, B, C\) and \(\delta\) are real. The unitary matrix \(U\) which diagonalizes \(D\) by \(U^\dagger DU\) is obtained by writing \(D\) eigenvectors in columns. The eigenvalues of
\( D \) are given by

\[
\frac{1}{2} \left( A + C \pm \sqrt{(A - C)^2 + 4B^2} \right)
\]  
(4.52)

which lead to the following eigenvectors

\[
u = \begin{pmatrix} d e^{-i\delta} \\ de^{-i\delta} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} de^{i\delta} \\ -1 \end{pmatrix}
\]  
(4.53)

with \( d \) given by

\[
d = -\frac{1}{2B} \left( A - C \pm \sqrt{(A - C)^2 + 4B^2} \right)
\]  
(4.54)

for which two solutions arise because of the freedom we have to associate \( u \) and \( v \) to either eigenvalue. As there is a freedom upon the sign in front of the square root we can write that \( d \) is given by

\[
d = \frac{A - C}{2B} \left( \sqrt{1 + \frac{4B^2}{(A - C)^2}} - 1 \right)
\]  
(4.55)

with the appropriate sign convention. Finally the \( U \) matrix has to satisfy \( U^\dagger U = \mathbb{I} \) thus it is given by

\[
U = \frac{1}{\sqrt{1 + d^2}} \begin{pmatrix} d e^{-i\delta} & de^{i\delta} \\ de^{-i\delta} & -1 \end{pmatrix}
\]  
(4.56)

One can remark that \( U \) is hermitian just as the matrix it diagonalizes. By identifying terms between the eigenvalues and \( d \) one can find that their are now given by the following relations

\[
U^\dagger D U = \begin{pmatrix} A + Bd & 0 \\ 0 & C - Bd \end{pmatrix}
\]  
(4.57)

One can also check this result explicitly by computing the complete unitary transformation \( U^\dagger D U \). Applying this result to the \( A_{\text{kin}} \) matrix defined in (4.50) which can be written as

\[
A_{\text{kin}} = \begin{pmatrix} 1 - a_{11} & -|a_{12}| e^{i\delta} \\ -|a_{12}| e^{-i\delta} & 1 - a_{22} \end{pmatrix}
\]  
(4.58)

yields to the following expression for \( d \)

\[
d = \frac{a_{11} - a_{22}}{2|a_{12}|} \left( \sqrt{1 + \frac{4|a_{12}|^2}{(a_{11} - a_{22})^2}} - 1 \right)
\]  
(4.59)

Let’s call \( U_1 \) the matrix diagonalizing \( A_{\text{kin}} \) with \( d \) given by (4.59). Then according to (4.57) we have

\[
U_1^\dagger A_{\text{kin}} U_1 = \begin{pmatrix} 1 - a_{11} - |a_{12}| d & 0 \\ 0 & 1 - a_{22} + |a_{12}| d \end{pmatrix} \equiv \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]  
(4.60)
We now introduce a stretching matrix $S$ so that there are no numerical factors in front of the kinetic terms of our Lagrangian. We define it as

$$S = \begin{pmatrix} \frac{1}{\sqrt{|\lambda_1|}} & 0 \\ 0 & \frac{1}{\sqrt{|\lambda_2|}} \end{pmatrix} \quad (4.61)$$

We now can write the kinetic part of our Lagrangian as

$$L_{\text{kin}} = i\overline{N}A_{\text{kin}}\partial N$$
$$= i\overline{NU}_1^\dagger A_{\text{kin}}U_1 \partial N$$
$$= i\overline{NU}_1 S^{-1} \left(SU_1^\dagger A_{\text{kin}}U_1 S^{-1}U_1 \right) \partial N$$
$$\equiv i\overline{N}U'N' \quad (4.62)$$

4.3.2 Mass Term

Having written the kinetic term in a canonical form we now have to rewrite the mass term of our Lagrangian. In a matrix form it is expressed as

$$M' = SU_1^\dagger \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} U_1 S$$
$$= \frac{1}{1 + d^2} \begin{pmatrix} \frac{M_1 + M_2 d^2}{|\lambda_1|} & \frac{(M_1 - M_2)de^{-i\delta}}{\sqrt{|\lambda_1 \lambda_2|}} \\ \frac{(M_1 - M_2)de^{-i\delta}}{\sqrt{|\lambda_1 \lambda_2|}} & \frac{M_1 d^2 + M_2}{|\lambda_2|} \end{pmatrix} \quad (4.63)$$

which is again of the form (4.51) with

$$\begin{cases} A' = \frac{M_1 + M_2 d^2}{(1 + d^2)|\lambda_1|} \\ B' = \frac{(M_1 - M_2) d}{(1 + d^2)\sqrt{|\lambda_1 \lambda_2|}} \\ C' = \frac{M_1 d^2 + M_2}{(1 + d^2)|\lambda_2|} \end{cases} \quad (4.64)$$

We can thus diagonalize it using the same procedure we used for the kinetic term. The unitary matrix diagonalizing $M'$ is given by

$$U_2 = \frac{1}{\sqrt{1 + r^2}} \begin{pmatrix} 1 & re^{i\delta} \\ re^{-i\delta} & -1 \end{pmatrix} \quad (4.65)$$

with $r$ given by

$$r = \frac{A' - C'}{2B'} \left(\sqrt{1 + \frac{4B'^2}{(A' - C')^2}} - 1\right) \quad (4.66)$$
From equation (4.57) we can express the masses eigentstates as $A' + B'r$ and $C' - B'r$.

### 4.3.3 $b$ Term

We now rotate the last term of our Lagrangian which will get a sign change when put into the Hamiltonian because it does not contain any derivatives. The matrix we have to compute is

$$b' = SU_1^† b U_1 S$$

where the $b$ matrix is

$$b = \begin{pmatrix} bK_{11} & bK_{12} \\ bK_{21} & bK_{22} \end{pmatrix} \equiv \begin{pmatrix} b_{11} & |b_{12}|e^{i\delta} \\ |b_{12}|e^{-i\delta} & b_{22} \end{pmatrix}$$

Then $b'$ is given by

$$\frac{1}{1 + d^2} S \begin{pmatrix} 1 & de^{i\delta} \\ de^{-i\delta} & -1 \end{pmatrix} \begin{pmatrix} b_{11} & |b_{12}|e^{i\delta} \\ |b_{12}|e^{-i\delta} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & de^{i\delta} \\ de^{-i\delta} & -1 \end{pmatrix} S$$

where $S$ is the streching matrix defined by (4.61) and $d$ the rotation parameter (4.59). Then by introducing the following notations

$$\begin{aligned} B_{11} &= \frac{1}{(1 + d^2)|\lambda_1|} (b_{22}d^2 + 2|b_{12}|d + b_{11}) \\
B_{12} &= \frac{1}{(1 + d^2)|\lambda_2|} (|b_{12}|d^2 + (b_{11} - b_{22})d - |b_{12}|) \\
B_{22} &= \frac{1}{(1 + d^2)|\lambda_2|} (b_{11}d^2 - 2|b_{12}|d + b_{22}) \end{aligned}$$

we can write $b'$ in a more compact form

$$b' = \begin{pmatrix} B_{11} & B_{12}e^{i\delta} \\ B_{12}e^{-i\delta} & B_{22} \end{pmatrix}$$

### 4.4 Summary

We can now write the Hamiltonian in the prime ('') basis.

$$H' = U_2 \tilde{H} U_2^† = U_2 \begin{pmatrix} \tilde{E}_1 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix} U_2^† - b'$$

where $\tilde{E}_i = \sqrt{k^2 + M_i^2}$ where the masses are respectively given by $A' + B'r$ and $C' - B'r$. Performing the $U_2$ rotation leads to

$$H' + b' = \frac{1}{1 + r^2} \begin{pmatrix} 1 & re^{i\delta} \\ re^{-i\delta} & -1 \end{pmatrix} \begin{pmatrix} \tilde{E}_1 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix} \begin{pmatrix} 1 & re^{i\delta} \\ re^{-i\delta} & -1 \end{pmatrix}$$

$$= \frac{1}{1 + r^2} \begin{pmatrix} \tilde{E}_1 + \tilde{E}_2 r^2 & (\tilde{E}_1 - \tilde{E}_2)re^{i\delta} \\ (\tilde{E}_1 - \tilde{E}_2)re^{-i\delta} & \tilde{E}_1 r^2 + \tilde{E}_2 \end{pmatrix}$$
As the Hamiltonian appears in a commutator in the QKE one can subtract any multiple of the unit matrix. We can thus write the equivalent Hamiltonian

\[ H' + b' = \frac{\Delta \tilde{E}}{1 + r^2} \begin{pmatrix} 0 & re^{i\delta} \\ re^{-i\delta} & r^2 - 1 \end{pmatrix} \] (4.74)

where we can approximate \( \Delta \tilde{E} \) by

\begin{align*}
\Delta \tilde{E} &= \sqrt{k^2 + \tilde{M}_2^2} - \sqrt{k^2 + \tilde{M}_1^2} \\
&= \sqrt{k^2 + (\tilde{M}_2 + \Delta \tilde{M})^2} - \sqrt{k^2 + \tilde{M}_2^2} \\
&\simeq \sqrt{k^2 + \tilde{M}_2^2} + 2\Delta \tilde{M} \tilde{M}_2 - \sqrt{k^2 + \tilde{M}_2^2} \\
&\simeq \sqrt{k^2 + \tilde{M}_2^2} \sqrt{1 + \frac{2\Delta \tilde{M} \tilde{M}_2}{k^2 + \tilde{M}_2^2}} - \sqrt{k^2 + \tilde{M}_2^2} \quad (4.75)
\end{align*}

Furthermore if \( a_{ij} \) and \( b_{ij} \) are small one can approximate the \( \lambda_i \) with one so that we can replace \( \Delta \tilde{E} \) by

\[ \Delta \tilde{E} \simeq \frac{\Delta M z}{\sqrt{k^2 + z^2}} \] (4.76)

In this case one can write

\[ H' \simeq \frac{\Delta M z}{(1 + r^2)\sqrt{k^2 + z^2}} \begin{pmatrix} 0 & re^{i\delta} \\ re^{-i\delta} & r^2 - 1 \end{pmatrix} - b' \] (4.77)

We can of course again subtract a multiple of the unit matrix in \( b' \) so that our Hamiltonian becomes

\[ H' \simeq \frac{\Delta M z}{(1 + r^2)\sqrt{k^2 + z^2}} \begin{pmatrix} 0 & re^{i\delta} \\ re^{-i\delta} & r^2 - 1 \end{pmatrix} - \begin{pmatrix} 0 & B_{12} e^{i\delta} \\ B_{12} e^{-i\delta} & B_{22} - B_{11} \end{pmatrix} \] (4.78)
5. DECA Y AND PRODUCTION RATES

5.1 Imaginary Part of the Self-Energy

We are now going to derive the imaginary part of the self energy and see how it is related to the production and decay rates. For this purpose we are going to compute the self-energy diagram with the Matsubara formalism developed in section 2.1. We have

\[
\Sigma = -K \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} S(\omega_n + \omega_a, \vec{p} + \vec{k}) D(\omega_n, \vec{p})
\]

\[
= -K \frac{1}{\beta} \sum_n \int \frac{d^3 p}{(2\pi)^3} S(\omega_n, \vec{p}) D(\omega_n - \omega_a, \vec{p} - \vec{k})
\]

where \(\omega_n\) and \(\omega_a\) are the fermionic Matsubara frequencies. According to (2.7) we have to evaluate the sum

\[
M = \frac{1}{\beta} \sum_n \frac{\omega_n \gamma^0 - \gamma \vec{p} + M_1}{\omega_n^2 - \vec{p}^2 - M_1^2} \frac{1}{(\omega_n - \omega_a)^2 - (\vec{p} - \vec{k})^2 - M_2^2}
\]

Defining \(E_1^2 = \vec{p}^2 + M_1^2\) and \(E_2^2 = (\vec{p} - \vec{k})^2 + M_2^2\) and using the residue theorem given by equation (2.14) our sum becomes

\[
M = -\sum \text{Res} \left[ \frac{z \gamma^0 - \gamma \vec{p} + M_1}{(z - E_1)(z + E_1)(z - \omega_a - E_2)(z - \omega_a + E_2)} \frac{1}{(\omega_a - E_1)^2 - (\vec{p} - \vec{k})^2 - M_2^2} \right]
\]

The two first poles are located in \(z = \pm E_1\) which will lead to hyperbolic tangent of \(E_1\beta/2\) which are related to the Fermi statistics by (2.13). The two other poles are located at \(z = \omega_a \pm E_2\) which leads to the hyperbolic cotangent of \(E_2\beta/2\) because

\[
\tanh \left( x + \frac{i \pi}{2} \right) = \coth(x)
\]

which is related to Bose-Einstein statistics through (2.13). One can find that the sum of residues is given by

\[
M = \frac{1}{2} \tanh \left( \frac{E_1 \beta}{2} \right) \frac{1}{2E_1} \left[ \frac{E_1 \gamma^0 - \gamma \vec{p} + M_1}{E_1^2 - (\omega_a - E_1)^2} + \frac{-E_1 \gamma^0 - \gamma \vec{p} + M_1}{E_1^2 - (\omega_a + E_1)^2} \right]
\]

\[
+ \frac{1}{2} \coth \left( \frac{E_2 \beta}{2} \right) \frac{1}{2E_2} \left[ \frac{(\omega_a + E_2) \gamma^0 - \gamma \vec{p} + M_1}{E_2^2 - (\omega_a + E_2)^2} + \frac{(\omega_a - E_2) \gamma^0 - \gamma \vec{p} + M_1}{E_2^2 - (\omega_a - E_2)^2} \right]
\]
The denominators can be cast into a more simple expressions with the use of
\[ \frac{1}{a^2 - b^2} = \frac{1}{2a} \left( \frac{1}{a + b} + \frac{1}{a - b} \right) \] (5.6)

We can now expand \( \mathcal{M} \) to the whole \( \omega \)-complex plane. As \( \mathcal{M}^* = \mathcal{M}(\omega^*) \) we express the discontinuity of \( \mathcal{M} \) as

\[
\text{Disc}\mathcal{M}(\chi) \equiv \lim_{\xi \to 0} \left[ \mathcal{M}(\chi + i\xi) - \mathcal{M}(\chi - i\xi) \right] = \lim_{\xi \to 0} \left[ \mathcal{M}^*(\chi - i\xi) - \mathcal{M}(\chi - i\xi) \right] = -2i \lim_{\xi \to 0} \text{Im}\mathcal{M}(\chi - i\xi) \] (5.7)

We have to compute the following limit
\[
\lim_{\eta \to 0} \left( \frac{1}{x + i\eta} - \frac{1}{x - i\eta} \right) \] (5.8)

which is equal to \( -2\pi i \delta(x) \) since one can define the Dirac delta distribution as
\[ \delta(x) = \frac{1}{\pi} \lim_{\eta \to 0} \frac{\eta}{x^2 + \eta^2}. \] (5.9)

It is now clear that if one uses (5.6) to simplify the denominator of \( \mathcal{M} \) and then computes the discontinuity of \( \mathcal{M} \) some \( \delta \) functions will appear. Finally, the discontinuity of \( \Sigma \) is obtained as

\[
\text{Disc}\Sigma = -iK \int \frac{d^3p}{(2\pi)^3} \frac{2\pi}{4E_1E_2} \left[ (\not{p} + M_1)(1 - n_F + n_B)\delta(\omega - E_1 - E_2) \\
+ (\not{p} + M_1)(n_F + n_B)\delta(\omega - E_1 + E_2) \\
+ (\not{p} - M_1)(n_F + n_B)\delta(\omega + E_1 - E_2) \\
+ (\not{p} - M_1)(1 - n_F + n_B)\delta(\omega + E_1 + E_2) \right] \] (5.10)

where of course \( n_F = n_F(E_1) \) and \( n_B = n_B(E_2) \). Moreover one can integrate the previous equation in which the kinetic term would have been evaluated for some spinor as \( \Pi = \bar{\pi}\Sigma u \) over \( d^3p_2 \) in order to obtain a \( \delta^4 \) distribution which would express energy-momentum conservation. In our case we want to consider the Majorana self-energy so the integrals involve integration over the Lepton and Higgs momentum

\[
\text{Im}\Pi = -\frac{\omega}{2\omega} \int \frac{d^3p_L}{2E_L(2\pi)^3} \int \frac{d^3p_\Phi}{2E_\Phi(2\pi)^3} (2\pi)^4 \left[ \delta^4(k - p_L - p_\Phi)|M|^2((1 - n_L)(1 + n_\Phi) + n_Ln_\Phi) \\
+ \delta^4(k + p_L - p_\Phi)|M|^2(n_L(1 + n_\Phi) + n_\Phi(1 - n_L)) \\
+ \delta^4(k - p_L + p_\Phi)|M|^2(n_\Phi(1 - n_L) + n_L(1 + n_\Phi)) \\
+ \delta^4(k + p_L + p_\Phi)|M|^2(n_Ln_\Phi + (1 - n_L)(1 + n_\Phi)) \right] \] (5.11)
5. Decay and Production Rates

which is a generalization of the Breit-Wigner distribution. One should note that we have added and subtracted some \( n_i n_j \) products in order to have a better interpretation of the previous equation. If the particle \( A \) is in the initial state its statistical weight is \( n \) independently of the nature of the particle. If it is in the final state the rate will be modified accordingly to the statistics of the particle. Fermions emission is suppressed by a \( 1 - n_F \) factor known as the Pauli blocking factor as boson emission is enhanced by a \( 1 + n_B \) factor which explains the spontaneous emission of photons for example.

5.2 Majorana Production Rate

Let us first consider processes that create Majorana Neutrino \( N \). First we compute their amplitude squared and then we will compute explicitly their rates. Finally, we will rescale all rates with respect to the temperature as performed in the previous chapter. Note that we do not include the coupling constants in the \( |M|^2 \) computation.

5.2.1 \(|M|^2\) Computation

\( L + \phi \rightarrow N \)

The \( M \) is given by

\[
M = i \bar{u}_N P_L u_L \tag{5.12}
\]

where \( P_L \) is the projector on the left chiral component and thus

\[
|M|^2 = \bar{u}_N u_L \bar{u}_L P_R u_N. \tag{5.13}
\]

When performing the spin sum we have to take the average over the spins of the incoming particles. Thus we have that

\[
|M|^2 = \frac{1}{4} \text{Tr}(\bar{p}_L p_N). \tag{5.14}
\]

As the trace of an odd number of \( \gamma \) matrices vanishes and \( \gamma \) matrices obey the Clifford algebra we finally find that

\[
|M|^2 = p_L \cdot p_N \tag{5.15}
\]

Using the 4-momentum conservation we can write

\[
p_L + p_\phi = p_N \tag{5.16}
\]

Recalling that the leptons are massless and putting \( p_L \) on the RHS and squaring the previous equation we have

\[
M^2_\phi = M^2 - 2p_N \cdot p_L \tag{5.17}
\]

By inserting the previous result in equation (5.15) we obtain

\[
|M|^2 = \frac{1}{2}(M^2 - M^2_\phi) \tag{5.18}
\]
5. Decay and Production Rates

\( \phi \rightarrow \bar{L} + N \)

We apply exactly the same procedure as in the previous reaction. \( M \) is now given by

\[
M = i \bar{u}_N P_L v_L
\]  

(5.19)

When taking the spin sum we do not have to perform any average because the only incoming particle is a spin-0 boson. Thus

\[
|M|^2 = \frac{1}{2} \text{Tr}(\phi_L \phi_N) = 2 F^2 p_L \cdot p_N
\]  

(5.20)

Using the 4-momentum conservation as it was previously already done we obtain

\[
|M|^2 = M_{\phi}^2 - M^2
\]  

(5.21)

5.2.2 \( \Gamma^p \) Computation

\( L + \phi \rightarrow N \)

From (5.11) we read that this process is described by the following rate

\[
\Gamma^p_1 = \frac{1}{2 \omega} \int \frac{d^3 p_L}{2E_L(2\pi)^3} \frac{d^3 p_\phi}{2E_\phi(2\pi)^3} |M|^2 (2\pi)^4 \delta^4(p_L + p_\phi - p_N) \cdot n_L n_\phi
\]  

(5.22)

which can be written as

\[
\Gamma^p_1 = \frac{|M|^2}{32 \pi^2 \omega} \int \frac{d^3 p_L}{E_L} \frac{d^3 p_\phi}{E_\phi} \delta(E_L + E_\phi - \omega) \delta^3(\vec{p}_L + \vec{p}_\phi - \vec{k}) \cdot n_L n_\phi
\]  

(5.23)

We can now evaluate the \( \delta^3 \)

\[
\Gamma^p_1 = \frac{|M|^2}{32 \pi^2 \omega} \int \frac{d^3 p_L}{E_L E_\phi} \delta(E_L + E_\phi - \omega) \cdot n_L n_\phi
\]  

(5.24)

where we have to remember that

\[
E_\phi = \sqrt{M_{\phi}^2 + (\vec{k} - \vec{p}_L)^2}
\]  

(5.25)

As there are no more 4-vectors in this section we will use the following notation \( p_L = |\vec{p}_L| \). We can now write

\[
E_L^2 = p_L^2 \rightarrow E_L dE_L = p_L dp_L
\]  

(5.26)

and thus

\[
d^3 p_L = p_L^2 dp_L d\Omega = p_L E_L dE_L d\Omega
\]  

(5.27)

where \( d\Omega = \sin \theta d\theta d\phi = 2\pi \sin \theta d\theta \) in our case. Then the integral becomes

\[
\Gamma^p_1 = \frac{|M|^2}{16 \pi^2 \omega} \int d\theta dE_L \sin \theta \frac{p_L}{E_\phi} \delta(E_L + E_\phi - \omega) \cdot n_L n_\phi
\]  

(5.28)
We are now going to use the fact that

$$\delta(f(\theta)) = \sum_{\theta_0} \frac{\delta(\theta - \theta_0)}{|f'(\theta_0)|}$$  \hspace{1cm} (5.29)$$

where \(f\) stands for \(E_L + E_\phi - \omega\) and where \(\theta_0\) is a zero of \(f\). From (5.25) we can write

$$E_\phi = \sqrt{M_\phi^2 + \vec{k}^2 + \vec{p}_L^2 - 2\vec{k}\vec{p}_L}$$  \hspace{1cm} (5.30)$$

which is easily expressed as a function of the integration variables by remembering that the leptons are massless. Then

$$E_\phi = \sqrt{M_\phi^2 + \omega^2 - M^2 + E_L^2 - 2|\vec{k}|E_L \cos \theta}$$  \hspace{1cm} (5.31)$$

Using this we immediately obtain

$$\delta(f(\theta)) = \frac{E_\phi}{|\vec{k}|E_L \sin \theta_0} \delta(\theta - \theta_0)$$  \hspace{1cm} (5.32)$$

The integral then becomes

$$\Gamma^{(1)}_p = \frac{|M|^2}{16\pi\omega|\vec{k}|} \int d\theta dE_L \delta(\theta - \theta_0) \cdot n_L n_\phi$$  \hspace{1cm} (5.33)$$

which we will write

$$\Gamma^{(1)}_p = \frac{|M|^2}{16\pi\omega|\vec{k}|} \int dE_L n_L n_\phi$$  \hspace{1cm} (5.34)$$

where we have to remember that

$$n_\phi = \frac{1}{e^{(\omega - E_L)/T} - 1}$$  \hspace{1cm} (5.35)$$

$$n_L = \frac{1}{e^{E_L/T} + 1}$$  \hspace{1cm} (5.36)$$

where the first result can be easily understood from energy conservation. It can also be found by remembering that

$$E_\phi = E_\phi(\theta = \theta_0)$$  \hspace{1cm} (5.37)$$

Then by computing the zero of \(f\) and putting it in the expression (5.31) we get the same result. We now have to take this integral over \(E_L\). Let us first determine what are the limits of integration. Writing the 4-momentum conservation and squaring it leads to

$$p_\phi = p_N - p_L \rightarrow M_\phi^2 = M^2 - 2p_N \cdot p_L$$  \hspace{1cm} (5.38)$$
where $p_N \cdot p_L$ can be expressed as $\omega E_L - |\vec{k}|E_L \cos \theta$ because the Lepton is massless. Then the Lepton energy can be expressed as

$$E_L = \frac{M^2 - M_\phi^2}{2(\omega - |\vec{k}| \cos \theta)}$$  \hspace{1cm} (5.39)

From the last result the minimum and maximum energy of the Lepton are respectively given by

$$\min(E_L) = \frac{M^2 - M_\phi^2}{2(\omega + |\vec{k}|)} \quad \text{and} \quad \max(E_L) = \frac{M^2 - M_\phi^2}{2(\omega - |\vec{k}|)}$$  \hspace{1cm} (5.40)

Finally,

$$\Gamma_{(1)}^p = \frac{|M|^2}{16 \pi |\vec{k}|} \int dE_L n_L n_\phi$$  \hspace{1cm} (5.41)

where the limits are implicitly understood.

$\phi \rightarrow \bar{L} + N$

As in the previous section the production rate can be expressed as

$$\Gamma_{(2)}^p = \frac{1}{2\omega} \int \frac{d^3p_L}{2E_L(2\pi)^3} \frac{d^3p_\phi}{2E_\phi(2\pi)^3} |M|^2(2\pi)^4 \delta(4)(p_\phi - p_L - p_N) \cdot n_\phi(1 - n_L)$$  \hspace{1cm} (5.42)

By applying exactly the same procedure we finally have

$$\Gamma_{(2)}^p = \frac{|M|^2}{16 \pi |\vec{k}|} \int dE_L n_\phi(1 - n_L)$$  \hspace{1cm} (5.43)

where $E_\phi$ is given by $\omega + E_L$. Let’s now fix the limits of integration. As in the previous relation we write the 4-momentum conservation and square it

$$p_\phi = p_N + p_L \rightarrow M_\phi^2 = M^2 + 2p_N \cdot p_L$$  \hspace{1cm} (5.44)

where $p_N \cdot p_L$ is given by $\omega E_L - |\vec{k}|E_L \cos \theta$. Then

$$E_L = \frac{M_\phi^2 - M^2}{2(\omega - |\vec{k}| \cos \theta)}$$  \hspace{1cm} (5.45)

and

$$\min(E_L) = \frac{M_\phi^2 - M^2}{2(\omega + |\vec{k}|)} \quad \text{and} \quad \max(E_L) = \frac{M_\phi^2 - M^2}{2(\omega - |\vec{k}|)}$$  \hspace{1cm} (5.46)

and thus

$$\Gamma_{(2)}^p = \frac{|M|^2}{16 \pi |\vec{k}|} \int dE_L n_\phi(1 - n_L)$$  \hspace{1cm} (5.47)

where the limits are implicitly understood.
5.3 Majorana Destruction Rate

5.3.1 $|M|^2$ Computation

$N \rightarrow L + \phi$

As before

$$M = i\bar{u}_L P_R u_N$$  \hspace{1cm} (5.48)

Performing the spin sum and taking the trace leads to

$$|M|^2 = p_L \cdot p_N$$  \hspace{1cm} (5.49)

Using the 4-momentum conservation the previous equation reads

$$|M|^2 = \frac{1}{2}(M^2 - M^2_\phi)$$  \hspace{1cm} (5.50)

$N + \bar{L} \rightarrow \phi$

Finally

$$M = i\bar{v}_L P_R u_N$$  \hspace{1cm} (5.51)

Performing the spin sum (here a 1/4 prefactor appears since we have to take the average both for the Neutrino and the Lepton) and taking the trace leads to

$$|M|^2 = \frac{1}{2}p_L \cdot p_N$$  \hspace{1cm} (5.52)

Using the 4-momentum conservation the previous equation reads

$$|M|^2 = \frac{1}{4}(M^2 - M^2_\phi)$$  \hspace{1cm} (5.53)

5.3.2 $\Gamma^d$ Computation

$N \rightarrow L + \phi$

From (5.11) we now read the destruction rate is given by

$$\Gamma^d_{(1)} = \frac{1}{2\omega} \int \frac{d^3 p_L}{2E_L(2\pi)^3} \frac{d^3 p_\phi}{2E_\phi(2\pi)^3} |M|^2 (2\pi)^4 \delta^4(p_N - p_L - p_\phi) \cdot (1 - n_L)(1 + n_\phi)$$  \hspace{1cm} (5.54)

which can be expressed as

$$\Gamma^d_{(1)} = \frac{|M|^2}{16\pi\omega |k|} \int dE_L (1 - n_L)(1 + n_\phi)$$  \hspace{1cm} (5.55)

where $E_\phi$ is given by $\omega - E_L$. We now have to fix the limits of integration. As usual we write the 4-momentum conservation and square it

$$p_\phi = p_N - p_L \rightarrow M^2_\phi = M^2 - 2p_N \cdot p_L$$  \hspace{1cm} (5.56)
and thus by expressing $p_N \cdot p_L$ as $\omega E_L - |\vec{k}| E_L \cos \theta$ we obtain

$$\min(E_L) = \frac{M^2 - M^2_{\phi}}{2(\omega + |k|)} \quad \text{and} \quad \max(E_L) = \frac{M^2 - M^2_{\phi}}{2(\omega - |k|)}$$

(5.57)

and thus

$$\Gamma_{d}^{(1)} = \frac{|M|^2}{16 \pi \omega |k|} \int dE_L (1 - n_L)(1 + n_{\phi})$$

(5.58)

where the limits are implicitly understood.

$N + \bar{L} \rightarrow \phi$

Finally

$$\Gamma_{d}^{(2)} = \frac{1}{2 \omega} \int \frac{d^3 p_L}{2E_L(2\pi)^3} \frac{d^3 p_{\phi}}{2E_{\phi}(2\pi)^3} |M|^2 (2\pi)^4 \delta^{(4)}(p_N + p_L - p_{\phi}) \cdot n_L(1 + n_{\phi})$$

(5.59)

can be expressed as

$$\Gamma_{d}^{(2)} = \frac{|M|^2}{16 \pi \omega |k|} \int dE_L n_L (1 + n_{\phi})$$

(5.60)

where $E_{\phi}$ is given by $\omega + E_L$. Let’s fix the limits of integration applying the same procedure we have used so far

$$p_{\phi} = p_N + p_L \rightarrow M_{\phi}^2 = M^2 + 2p_N \cdot p_L$$

(5.61)

with $p_N \cdot p_L = \omega E_L - |\vec{k}| E_L \cos \theta$ so that

$$\min(E_L) = \frac{M_{\phi}^2 - M^2}{2(\omega + |k|)} \quad \text{and} \quad \max(E_L) = \frac{M_{\phi}^2 - M^2}{2(\omega - |k|)}$$

(5.62)

and

$$\Gamma_{d}^{(2)} = \frac{|M|^2}{16 \pi \omega |k|} \int dE_L n_L (1 + n_{\phi})$$

(5.63)

where the limits are implicitly understood.

### 5.4 Summary and Rescaling

We here collect the rates we computed with the same approximation of massless Higgs we have made when computing the Hamiltonian. The production of Majorana is then only possible by the Lepton plus Higgs to Majorana process. Its rate is given by

$$\Gamma^p = \frac{M^2}{32 \pi \omega k} \int dE_L n_L n_{\phi}$$

(5.64)
We can perform the following change of variables $x = \frac{E}{L}$ which gives

$$\Gamma^p = \frac{M^2 T}{32\pi \omega k} \int dx n_L n_\phi$$

(5.65)

where the limits are of course also rescaled by $1/T$. Performing the rescaling on the momentums and thus on $\omega$ and introducing $z$ leads to

$$\Gamma^p = \frac{Mz}{32\pi \omega k} \int dx \frac{1}{e^x + 1} \frac{1}{e^{\omega - x} - 1}$$

(5.66)

Using the approximative dispersion relation gives us the final form of the production rate

$$\Gamma^p = \frac{Mz}{32\pi \sqrt{k^2 + z^2} k} \int dx \frac{1}{e^x + 1} \frac{1}{e^{\sqrt{k^2 + z^2} - x} - 1}$$

(5.67)

where the integral over $x$ has to be performed between

$$x_{\text{min}} = \frac{z^2}{2(\sqrt{k^2 + z^2} + k)} \quad \text{and} \quad x_{\text{max}} = \frac{z^2}{2(\sqrt{k^2 + z^2} - k)}$$

(5.68)

The destruction of Majorana particles is only made possible by its decay into a Lepton and a Higgs particle. Its rescaled expression is obtained by performing the same change of variables as in the production case and gives

$$\Gamma^d = \frac{Mz}{32\pi \sqrt{k^2 + z^2} k} \int dx \left(1 - \frac{1}{e^x + 1}\right) \left(1 + \frac{1}{e^{\sqrt{k^2 + z^2} - x}} - 1\right)$$

(5.69)

where the limits of integration are the same as the ones for $\Gamma^p$. 
6. SOLVING THE QUANTUM KINETIC EQUATION

In the last chapters we have computed all elements entering the Quantum Kinetic Equation, namely the Hamiltonian and the Production and Decay Rates of the Majorana particles. They are respectively given by equations (4.78), (5.67) and (5.69). The QKE itself has the following form

\[ iM^2 \frac{d\rho}{zM_0 dz} = [\rho, \mathcal{H}] - i \frac{1}{2} \{(\Gamma^d + \Gamma^p)K, \rho - \rho_{eq}\} \]  

(6.1)

in which we dropped the \( k \) subscript which is now implicit. To proceed further we have to explicit the equation for \( \rho \) which is an 8 by 8 matrix in our case. The complexity of this system does not allow us to solve it exactly. Let us remind the situation. We consider two flavours of light neutrinos of which one is thermalized. The initial condition are on \( \rho \)'s are thus

\[ \rho_N(z_i) = 0 \quad \rho_{L1}(z_i) = 0 \quad \rho_{L2}(z_i) = e^{-k} \]  

(6.2)

Then both Majorana neutrinos are created due to the initial abundance of \( L_2 \). These Majorana neutrinos then decay with a CP-asymmetry and generate both \( L_i \)'s. The \( L_2 \) lepton will then be able to couple to the Higgs and produce a Majorana neutrino. This process is called back-reaction process. We will neglect back-reaction process due to the \( L_1 \) lepton since its rate is negligible when the \( L_1 \) population become significant.

In order to estimate the behaviour of the number densities we now assume \( \rho_N \) obeys the following equation

\[ iM^2 \frac{d\rho_N}{zM_0 dz} = [\rho_N, \mathcal{H}] - i \frac{1}{2} \{(\Gamma^d + \Gamma^p)K, \rho_N - \rho_{eq}^{\rho_N}\} \]  

(6.3)

where \( \rho_{eq}^{\rho_N} \) is determined by solving

\[ \{K(\Gamma^d + \Gamma^p), \rho_{eq}^{\rho_N}\} = 2K\Gamma^p \]  

(6.4)

where the factor two comes from the anticommutator of \( \Gamma^p \) with the unit matrix. The lepton asymmetry generated by the CP-violating decay of the Majorana neutrino has now to be computed. One should here solve the QKE for...
6. Solving the Quantum Kinetic Equation

\( \rho \)'s. In first approximation let us assume only \( L_1 \) generates the asymmetry. Then the QKE for \( \rho_L \) is the same as the Boltzmann Equation since the commutator vanishes and anticommutators pick a factor two. Then the evolution of the leptonic asymmetry can be expressed [BDB06] as

\[
\frac{d\rho_{\Delta}}{dz} = \frac{\epsilon^{CP} K_{11} \Gamma^d}{H(z = 1) z} (\rho_N - \rho_{\Delta}^{eq})
\]

(6.5)

where \( \epsilon^{CP} \) can be read from [ABP06] as

\[
\frac{\text{Im}(K_{12}^2)}{8\pi K_{11}} \frac{M_1 M_2 (M_2^2 - M_1^2)}{\left( M_2^2 - M_1^2 - \frac{1}{\pi} M_2 K_{22} \Gamma^d \log \left( \frac{M_2^2}{M_1^2} \right) \right)^2 + (M_2 K_{22} \Gamma^d - M_1 K_{11} \Gamma^d)^2}
\]

(6.6)

which can be greatly simplified since \( M_1 \approx M_2 \). The logarithmic terms thus can be set to zero. Difference between \( M_1 \) and \( M_2 \) have to be left as is while \( M_1 + M_2 \) can safely be put to \( 2M \). Defining \( \Delta \) as \( \Delta M/M \) we get

\[
\epsilon^{CP} \approx \frac{2\Delta}{4\Delta^2 + \left( \frac{K_{22} - K_{11}}{16\pi} \right)^2}
\]

(6.7)

which is valid as long as

\[
\Delta \gg \frac{1}{16\pi} |K_{22} - K_{11}|
\]

(6.8)

Furthermore the \( H(z = 1) \) factor simply gives rise to

\[
H = \frac{T^2}{M_0} = \frac{M^2}{z^2 M_0} \rightarrow H(z = 1) = \frac{M^2}{M_0}
\]

(6.9)

which come from the \( t \rightarrow z \) variable change and where \( M_0 \) is given by (3.8).

Let us now present some numerical results where we have set \( K_{12}, K_{21} \ll K_{11}, K_{22} \). As we have assumed only \( L_1 \) produces leptonic asymmetry and neglected back-reactions from \( L_1 \) we have set the production due to \( L_1 \) to zero. Figure 6.1 illustrates how the density of Majorana particle \( N_1 \) evolves with \( z \) for different values of its momentum. We fixed its initial abundance to zero and see that it has the awaited behaviour as they tend to disappear when \( z \) or time grows. We can measure the effect of the presence of the Hamiltonian in the QKE by setting it to zero. The difference between QKE with and without Hamiltonian is numerically null for the set of parameter we have chosen even when modifying the \( \Delta \) parameter. Modifying \( \Delta \) nevertheless modifies the \( \rho_\Delta \) prediction significantly. One has to be careful to choose parameters such that the 6.8 is always satisfied.

On figure 6.2 we present the Lepton asymmetry generated by our model for the same values of \( \epsilon \). Note that this plot is in a log-log scale. This figure shows that our model predicts a leptonic freeze-out which \( \text{via} \) sphalerons processes...
Fig. 6.1: Evolution of $\rho_N$ in function of $z$. The different lines correspond to $k$ from 0 to 5 with a 0.5 step. Lesser the $k$ the higher the density.

can be converted into baryonic asymmetry.

The study of the Majorana density in function of $k$ for different values of $z$ is shown on figure 6.3. The average behaviour can be localized at $\sim 3$ which can be compared with the momentum thermal average of a Fermi-Dirac distribution which is given by

$$\langle p \rangle_F = \int \frac{d^3p}{(2\pi^3)} \frac{p}{e^p + 1} \left( \int \frac{d^3q}{(2\pi^3)} \frac{1}{e^q + 1} \right)^{-1} = \frac{7\pi^4}{180\zeta(3)} \simeq 3.15 \quad (6.10)$$

This model gives satisfying results and can generate suitable $\Delta L$ which is the integral over all modes of $\rho_\Delta$ for leptogenesis. Nevertheless we have used approximate equations and neglected some contributions to be able to have at least an idea of the evolution of number densities of RH neutrinos and leptons.
Fig. 6.2: Evolution of $\rho_\Delta$ in function of $z$. The different can be identified with the help of figure 6.1.

Fig. 6.3: Evolution of $\rho_N$ in function of $k$. The average momentum of a Fermi-Dirac distribution is also plotted for comparison.
7. CONCLUSION

In this paper we have studied the system of two degenerate RH neutrinos coupled to light leptons via see-saw type Yukawa interaction. Using the FTFT formalism we have computed both Hamiltonian and rates of production and destruction for the Majorana particle. The aim of this work was to use the QKE to estimate the lepton asymmetry Majorana neutrinos can create through its CP-violating decay. For that purpose we have developed all necessary routines.

We have applied the above to a simple sample case where, first, we have neglected the Higgs decay and, therefore, an additional CP-violating source. Second, we have made some assumptions about how the densities of Majorana particles and Leptons were related. In the equations we presented in this paper the Majorana neutrino density obeys an equation where all the destruction and production rates were computed as if Leptons were in thermal equilibrium. This is, of course, is not a realistic case and departure of Leptons from thermal equilibrium has to be taken into account.

Further investigations will be made in order to write coupled equations for Majorana neutrinos and Leptons as it was made in [AS05] for example. We are currently working in this direction and beginning to have interesting results which will be reported elsewhere. The final goal will be to study the resonant leptogenesis scenario via QKE in the most realistic setup and to compare the outcome with the results reported in [PU04].
APPENDIX
A. MAJORANA FIELDS

In this section we recall a certain number of facts about Majorana neutrinos, in particular how to construct a Lagrangian with such fields. Let us define the Dirac field $\psi$ as

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \eta \end{pmatrix} \equiv \psi_R + \psi_L$$ (A.1)

where $\xi$ and $\eta$ are Weyl fields. The kinetic term of a Dirac Lagrangian is given by $i \overline{\psi} \gamma^\mu \partial_\mu \psi$ which gives in terms of left and right components

$$i \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + i \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R + i \overline{\psi}_R \gamma^\mu \partial_\mu \psi_L$$ (A.2)

where the two last terms vanish by construction. The most general Majorana condition requires that the charge conjugation of a field $\chi$ is identical to $\chi$ up to a phase. In particular

$$\chi = e^{i\alpha} \psi_R + e^{i\beta} \psi_R^c$$ (A.3)

satisfies the Majorana condition since $\chi^c = e^{-i(\alpha + \beta)} \chi$. We now would like to construct a Lagrangian in terms of the $\chi$ field. We first focus on the kinetic term. As we have no left-handed field in our theory the kinetic term reduces to $i \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R$. As

$$\psi_R = e^{-i\alpha} \frac{1 + \gamma^5}{2} \chi$$ (A.4)

we can write the kinetic term as

$$i \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R = i e^{i\alpha} \frac{1 + \gamma^5}{2} \chi \partial_\mu e^{-i\alpha} \frac{1 + \gamma^5}{2} \chi = \frac{i}{2} \overline{\chi} \phi \chi + \frac{i}{2} \overline{\chi} \gamma^5 \chi$$ (A.5)

The first term is the term we could have expected. The second one is more problematic. Let us explicit its terms as

$$\overline{\chi} \gamma^5 \chi = (e^{-i\alpha} \overline{\psi}_R + e^{-i\beta} \overline{\psi}_R^c) \gamma^5 (e^{i\alpha} \psi_R + e^{i\beta} \psi_R^c)$$ (A.6)

The terms involving one $\psi_R$ and one $\psi_R^c$ field vanish because of the charge conjugation which inverts the left and right Weyl fields. Indeed as terms like $\overline{\psi}_R \gamma^5 \psi_R$ do not vanish it is straightforward to see that $\overline{\psi}_R \gamma^5 \psi_R$ and $\overline{\psi}_R \gamma^5 \psi_R^c$
do. Then the two remaining terms are

\[ \bar{\psi}_R \gamma^5 \psi_R = \begin{pmatrix} 0 & \xi^\dagger \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_\mu \eta \\ 0 \end{pmatrix} = \xi^\dagger \sigma^\mu \partial_\mu \xi \] (A.7)

and

\[ \bar{\psi}_c \gamma^5 \psi_c = \begin{pmatrix} 0 & \xi^\dagger \\ \bar{\sigma}^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -i \sigma^2 \partial_\mu \xi^* \end{pmatrix} = \xi^\dagger \sigma^2 \sigma^\mu \partial_\mu \xi^* \] (A.8)

In the last step we used the \((\sigma^2 \sigma^\mu \sigma^2)^* = \sigma^\mu\) identity and the fact that \(\partial_\mu \eta^* = (\partial_\mu \eta)^*\). As we can notice the sum of the two terms vanishes so that the kinetic term written in function of \(\chi\) is simply

\[ \frac{i}{2} \nabla \phi \chi \] (A.9)

If we now would like to add a mass term the Lagrangian would become of the following form

\[ \frac{i}{2} \nabla \phi \chi - \frac{m}{2} \nabla \chi \] (A.10)

We can of course now express this Lagrangian in terms of the \(\psi_R\) field. The kinetic term is of course \(i \bar{\psi}_R \gamma^5 \psi_R\). The mass term is given by

\[ \frac{m}{2} (e^{-i \alpha} \bar{\psi}_R + e^{-i \beta} \bar{\psi}_c)(e^{i \alpha} \psi_R + e^{i \beta} \psi_c) \] (A.11)

in which the two terms \(\bar{\psi}_R \psi_R\) and \(\bar{\psi}_c \psi_c\) vanish identically. By defining \(M\) as \(me^{i(\alpha - \beta)}\) and noticing that \((\bar{\psi}_c \psi_R)^* = -\bar{\psi}_R \psi_c\) we can write the Majorana Lagrangian as

\[ i \bar{\psi}_R \gamma^5 \psi_R - \frac{M}{2} \bar{\psi}_R \psi_R + \text{h.c.} \] (A.12)
B. TRACES COMPUTATION

In this section we will perform a detailed computation of the integrals appearing in equations (4.15) to (4.17).

\[
\frac{1}{4} \text{Tr(Re}\Sigma^\prime) = K M_1 \int \frac{d^4 p}{(2\pi)^4} \left[ \frac{\Gamma_B(p)}{(p + K)^2 - M^2} - \frac{\Gamma_F(p + K)}{p^2 - M^2} \right]
\]  

(B.1)

Replacing the \( \Gamma \) functions by their definition we obtain

\[
K M_1 \int \frac{d^4 p}{(2\pi)^3} \left[ \frac{\delta(p^2 - M^2) n_B(|p \cdot u|)}{(p + K)^2 - M^2} - \frac{\delta((p + K)^2 - M^2) n_F(|p + K \cdot u|)}{p^2 - M^2} \right]
\]  

(B.2)

In order to simplify the second term we shift \( p + K \rightarrow p \) and get

\[
K M_1 \int \frac{d^4 p}{(2\pi)^3} \left[ \frac{\delta(p^2 - M^2) n_B(|p \cdot u|)}{p^2 + K^2 + 2p \cdot K - M^2} - \frac{\delta((p + K)^2 - M^2) n_F(|p + K \cdot u|)}{(p - K)^2 - M^2} \right]
\]  

(B.3)

The expression of the two terms is now essentially the same. We will thus only compute the first one. It can be written

\[
K M_1 \int \frac{d^4 p}{(2\pi)^3} \left[ \frac{\delta(p_0^2 - |\vec{p}|^2 - M^2) n_B(|p \cdot u|)}{p^2 + K^2 + 2p \cdot K - M^2} \right]
\]  

(B.4)

where we can immediately replace \( p^2 \) by \( M^2 \) because of the delta-function. Developing \( d^4 p \) yields

\[
K M_1 \int \frac{d^4 p}{(2\pi)^3} \left[ \frac{\delta(p_0^2 - |\vec{p}|^2 - M^2) n_B(|p \cdot u|)}{M^2 - M^2 + K^2 + 2p_0 K_0 - 2|\vec{p}||\vec{K}| \cos \theta} \right]
\]  

(B.5)

From the \( k \) and \( \omega \) properties we discussed in chapter 4 we have \( K_0 = \omega \) and \( |\vec{K}| = k \). Using the property

\[
\delta(f(x)) = \sum_{x_0} \frac{\delta(x - x_0)}{|f'(x_0)|}
\]  

(B.6)

where \( x_0 \) are the zeros of \( f \) we have that

\[
\delta(p_0^2 - |\vec{p}|^2 - M^2) = \frac{1}{2\epsilon_2} \left[ \delta(p_0 - \epsilon_2) + \delta(p_0 + \epsilon_2) \right]
\]  

(B.7)
with \( \epsilon_i = \sqrt{|p|^2 + M_i^2} \). Then the integral over \( p_0 \) leads to two terms which are computed in the heat bath rest frame

\[
KM_1 \int \frac{d|\vec{p}| \sin \theta d\theta |\vec{p}|^2}{2(2\pi)^2} \frac{n_B(\epsilon_2)}{\epsilon_2} \frac{1}{M_1^2 - M_1^2 + K^2 + 2\epsilon_2 \omega - 2|\vec{p}| |\vec{K}| \cos \theta} \nonumber
\]

\[
+ \frac{1}{M_2^2 - M_1^2 + K^2 - 2\epsilon_2 \omega - 2|\vec{p}| |\vec{K}| \cos \theta}
\]

We are now left with integrals over \( \theta \) which are of the form

\[
\int_0^\pi d\theta \sin \theta \frac{1}{a + b \cos \theta} = \frac{1}{b} \log \left( \frac{a + b}{a - b} \right)
\]

We can introduce the following notations \( p = |\vec{p}| \) and \( \Delta = M_2^2 - M_1^2 \). We now obtain

\[
KM_1 \int \frac{dp}{4(2\pi)^2} \frac{n_B(\epsilon_2) p}{k} \left[ \log \left( \frac{\omega^2 - k^2 + \Delta + 2\epsilon_2 \omega + 2pk}{\omega^2 - k^2 + \Delta + 2\epsilon_2 \omega - 2pk} \right) \right. \nonumber
\]

\[
+ \left. \log \left( \frac{\omega^2 - k^2 + \Delta - 2\epsilon_2 \omega + 2pk}{\omega^2 - k^2 + \Delta - 2\epsilon_2 \omega - 2pk} \right) \right]
\]

Almost the same computation has to be made for the second part. The computation of \( \frac{1}{4} \text{Tr}(K \Re \Sigma') \) is done by the same way except that we now have an extra \( K^2 - p \cdot K \) factor in the front of the delta-functions. This will lead to the following term in the numerator

\[
K^2 + p \cdot K = \omega^2 - k^2 + p_0 \omega - |\vec{p}| \omega \cos \theta
\]

and thus the integration over the solid angle will be of the form

\[
\int_0^\pi d\theta \sin \theta \frac{a - b \cos \theta}{c - 2b \cos \theta} = \frac{1}{4b} \left[ 4b + (2a - c) \log \left( \frac{c + 2b}{c - 2b} \right) \right]
\]

The third trace \( \frac{1}{4} \text{Tr}(\Re \Sigma') \) is also computed in the heat bath rest frame so that

\[
u \cdot p + u \cdot K = p_0 + \omega
\]

and the integration over the solid angle is of the same form as in the first case. To express the result of these computations in a more compact way it is convenient to define the following logarithmic functions ([We82],[QVC95])

\[
L^\pm_1 = \log \left( \frac{\omega^2 - k^2 - \Delta - 2\epsilon_1 \omega - 2kp}{\omega^2 - k^2 - \Delta - 2\epsilon_1 \omega + 2kp} \right) \pm \log \left( \frac{\omega^2 - k^2 - \Delta + 2\epsilon_1 \omega - 2kp}{\omega^2 - k^2 - \Delta + 2\epsilon_1 \omega + 2kp} \right)
\]
\[ L_2^\pm = \log \left( \frac{\omega^2 - k^2 + \Delta + 2\epsilon_2 \omega + 2kp}{\omega^2 - k^2 + \Delta + 2\epsilon_2 \omega - 2kp} \right) \pm \log \left( \frac{\omega^2 - k^2 + \Delta - 2\epsilon_2 \omega + 2kp}{\omega^2 - k^2 + \Delta - 2\epsilon_2 \omega - 2kp} \right) \]  

(B.15)

Finally one obtains

\[
\frac{1}{4} \text{Tr}(\text{Re}\Sigma') = KM \int \frac{dp}{16\pi^2 k} \left( \frac{n_B(\epsilon_2) L_2^+}{\epsilon_2} + \frac{n_F(\epsilon_1) L_1^+}{\epsilon_1} \right) \]  

(B.16)

\[
\frac{1}{4} \text{Tr}(K\text{Re}\Sigma') = K \int \frac{dp}{16\pi^2 k} \left[ \frac{n_B(\epsilon_2) L_2^+}{\epsilon_2} + \frac{n_F(\epsilon_1) L_1^+}{\epsilon_1} \right] \]  

(B.17)

\[
\frac{1}{4} \text{Tr}(\gamma\text{Re}\Sigma') = K \int \frac{dp}{16\pi^2 k} \left[ n_B(\epsilon_2) \left( L_2^- + \frac{\omega}{\epsilon_2} L_2^+ \right) + n_F(\epsilon_1) L_1^- \right] \]  

(B.18)

where

\[
L_i^+ = 4pk + \frac{\omega^2 - k^2 - \Delta}{2} L_i^+ \]  

(B.19)
References


