Robert J. ADLER (**Technion Haïfa**)

Gaussian random fields on manifolds

This talk will be rather more of a propaganda piece than is commonly the case and is planned is to encourage you to take a serious look at the study of Gaussian random fields on Riemannian manifolds. In an attempt to convince you that this is worthwhile (beyond just being a lot of fun), the talk will have five parts to it. It should also be more user friendly than this "extended abstract."

1. Gaussian processes and Riemannian manifolds. Centered Gaussian processes $f : T \to \mathbb{R}$ are easily defined on any topological space T via the covariance function $C(s,t) = \mathbb{E}\{f(s)f(t)\} : T \times T \to \mathbb{R}$, and require no structural information on T. This makes them perfect for models of processes on manifolds.
N-manifolds are defined via local diffeomorphisms from an atlas of coverings to $\mathbb{R}^N$. In other words, an N manifold is something that looks locally like $\mathbb{R}^N$. For examples of 2 manifolds embedded in $\mathbb{R}^3$, think of the sphere S^2 and the torus T^2. Cut them in half and you have manifolds with boundary. C^k-manifolds come with tangent spaces T_tM at each $t \in M$. Elements of T_tM are written as X_t, and can be thought of as simple tangent vectors to the surface M at t or, in proper generality, as differential operators, so that $(X_tf)(t)$ is the "derivative in the direction X_t of the function $f\ M \to \mathbb{R}$, at the point $t \in M$".
Sometimes manifolds also have an associated "Riemannian metric" g_t, which for each $t \in M$, is a real valued, positive definite function on $T_tM \times T_tM$. Among other things, g allows the definition of *geodesic distance* in M. The pair (M,g) is called a *Riemannian manifold*. Here is one way to define a Riemannian metric:

$$g_t(X_t,Y_t) \triangleq \mathbb{E}\{(X_tf)(t)\ (Y_tf)(t)\} \tag{1}$$

where $X_t,Y_t \in T_tM$. We call g the *metric induced by* the Gaussian field f. Note that this metric "knows" about two things: it knows about the underlying geometry of M through the tangent spaces T_tM, and it knows about the random field f through the expectation in (1).

2. Some applications. Of these some are "real", some to other areas of mathematics, some already applied and some still at the planning stage. At the end of the talk you will be expected to suggest more of your own.
I know of two classes of examples in which Gaussian processes arise on physical manifolds. The first, chronologically, arose in the study of the anomalies in the cosmic microwave background (CMB) radiation, a sort of signature left over from the creation of the universe (cf. [7, 11].) The data in this study is directional, representing radiation coming into the point of measurement from the surrounding universe. Taking a standard signal plus noise model, with coloured Gaussian noise, one ends up with a random field f on the two-dimensional sphere S^2. Astronomers use a statistical hypothesis test based on the statistic

$$\sup_{t\in S^2} f(t). \tag{2}$$

The second example comes from bio-statistics, and is related to brain imaging. I refer you to Keith Worsley's talk and paper for details. While in most of the examples here the manifold is a subset of flat Euclidean space, recent extensions to mapping the cortical surface (rather than the interior of the brain) require far more sophisticated manifolds, with dimensions as high as six[1]. Once again, the statistic that arises in the analysis of cortical data is, as in (2), a supremum, although now over a higher dimensional manifold, with boundary.
Here is an example whose applications are not as immediate: let $A = (a_{ij})$ be a symmetric, $N \times N$ matrix of Gaussian variables. The largest eigenvalue of A is

$$\lambda_{\max} \triangleq \sup_{t\in S^{N-1}} tAt' \equiv \sup_{t\in S^{N-1}} f(t),$$

where $f(t) = tAt'$ is a centered Gaussian process on S^{N-1}. Thus, the supremum of f is again a source of interesting information.

1. Why 6? Roughly: Measurements are taken over a two dimensional cortical surface, averaged over the projection onto the cortical surface (via the standard exponential map, say) of ellipses in the tangent plane, the two axes of which are allowed to vary and which are rotated through two dimensions. 2+2+2=6.

For the last example, take *exponential families*, which lie at the core of modern mathematical statistics. Assuming an appropriate reference measure μ on $\mathbb{R}^N$, densities of N-dimensional random variables X coming from an exponential family with parameter $t \in \mathbb{R}^N$ are of the form $p(x|t) \exp\{xt' - \psi(t)\}$, for some ψ. The natural parameter space for t is

$$M = \left\{ t \in \mathbb{R}^N : \int \exp\{xt'\} \, p(x|t) \, \mu(dx) < \infty \right\},$$

which, in general, is a manifold. Use the mapping $t \to p(\cdot|t)$ to think of M as a manifold in function space and the elements of $T_t M$ as functions. The log-likelihood function $\ell : M \times \mathbb{R}^N \to \mathbb{R}$ is then $\ell(t)(x) = \log(p(x|t))$. There is a natural Riemannian metric on M, defined, for $Y_t, Z_t \in T_p M$, by

$$g_t(Y_t, Z_t) = \int (Y_t(\ell_t))(x) \, (Z_t(\ell_t))(x) \, p(x|t) \, \mu(dx).$$

This may look unfamiliar, but in fact all it does is extend the familiar Fisher information matrix to obtain a Riemannian metric on M. The geodesic distance induced by this g is called the *information metric*.
So far, so good. But where is the Gaussian process? Since we are talking about Statistics, it must come from large sample asymptotics, and this is indeed the case. Where is its supemum? That comes from the likelihood ratio test.

3. The Euler-Poincaré characteristic. Suppose $A \subset M$ is *triangulisable*, which basically means that it can be broken up into the finite union of N-dimensional sets, each one of which is the diffeormorphic image of an N-simplex. A k face of one of these sets is the image of a k face of the simplex.
Let $\alpha_k = \alpha_k(A)$ be the total number of k faces appearing in a triangulation of A. The *Euler-Poincaré characteristic* of A, which is independent of the triangulation, is

$$\varphi(A) = \alpha_0 - \alpha_1 + \cdots + (-1)^N \alpha_N. \tag{3}$$

Here are some simple examples. If $A \subset \mathbb{R}^1$ is made up of m disjoint closed intervals, then $\varphi(A)$ is simply m. If $A \subset \mathbb{R}^2$ is made up of m units homotopic to a disc, with a total of n holes in them, then $\varphi(A) = m - n$. If $A \subset \mathbb{R}^3$ is made up of m units homotopic to a baseball, with n handles stuck on them, and p holes placed inside them, then $\varphi(A) = m - n + p$.

4. The expected EP characteristic in a Gaussian setting. Here is the punchline, straight out of the McGill PhD. thesis [9] of Jonathan Taylor. Much of what is said here has been cut and pasted either from there or [10], and will reappear in [2].
The excursion set of f on M over the level $u \in \mathbb{R}$ is the set

$$A_u \equiv A_u(f, M) \triangleq \{t \in M \; f(t) \geqslant u\}, \tag{4}$$

If f is centered Gaussian with unit variance, and various smoothness conditions hold on f and M (assumed to have no boundary), then

$$\mathbb{E}\{\varphi(A_u(f, M))\} = \sum_{j=0}^{N} \mathcal{L}_j(M) \rho_j(u), \tag{5}$$

where

$$\mathcal{L}_j(M) = \begin{cases} (2\pi)^{-(N-j)/2} \int_M \left(\left(\frac{N-j}{2}\right)! \right)^{-1} \mathrm{Tr}(-R)^{(N-j)/2} \, \mathrm{Vol}_g & N - j \geqslant 0 \text{ is even,} \\ 0 & N - j \text{ is odd,} \end{cases}$$

are known as the Lipschitz-Killing curvatures of M, calculated with respect to the metric g induced by f and Vol_g is the volume form on M determined by g. R is the corresponding curvature tensor and Tr the trace operator on the algebra of double forms. Finally, and somewhat more simply, the so-called *Euler densities* ρ_j are given by

$$\rho_j(u) = \frac{1}{(2\pi)^{\frac{j+1}{2}}} \int_u^\infty H_j(t)\, \mathrm{e}^{-t^2/2}\, dt = \begin{cases} \frac{1}{(2\pi)^{\frac{j+1}{2}}} H_{j-1}(u)\, \mathrm{e}^{-u^2/2} & j \geqslant 1, \\ 1 - \Phi(u) & j = 0, \end{cases}$$

where $H_j(x)$ is the j-th Hermite polynomial and Φ is the cumulative distribution of a $N(0,1)$ variable. There is a similar result for manifolds with boundary, in which case the Lipschitz-Killing curvatures involve an extra integral over ∂M. Details, along with a long history of earlier results, can be found in [2, 9, 10]. Formula (5) is deep. But even if you do not care about depth, there are practical reasons to care about it.

5. The Euler characteristic and the tail of the supremum. One of reasons that (5) is so important is that the Euler characteristic of excursion sets is just about the only random variable associated with Gaussian process excursions for which an explicit, non-asymptotic expression is available for anything related to its distribution. Furthermore, it tells us a lot (asymptotically at least) about another random variable which we have already seen arising as central in a number of examples, for, under quite general conditions, there exists a constant $\alpha > 1$ such that, for all $u > 0$,

$$\left| \mathbb{P}\left\{ \sup_{t\in M} f(t) \geqslant u \right\} - \mathbb{E}\left\{ \varphi(A_u(f,M)) \right\} \right| \leqslant O\left(e^{-\alpha u^2/2} \right). \tag{6}$$

This result has been proven many times (although not always recognised as being in this form) for many special cases, including [3, 4, 5, 6, 8]. If you now relate this back to the examples described above, which all had to do with $\sup f$, you will understand why (5) is important.
An almost free result related to (6) is the fact (implicit in all the references above because of computations in [4]) that if $N_u(f,M)$ is either the number of local maxima, or number of extrema, of f above the level u over M, then

$$\left| \mathbb{E}\left\{ N_u(f,M) \right\} - \mathbb{E}\left\{ \varphi(A_u(f,M)) \right\} \right| \leqslant O\left(e^{-\alpha u^2/2} \right). \tag{7}$$

However, unlike $\mathbb{E}\{\varphi(A_u(f,M))\}$, there is no known closed form expression for $\mathbb{E}\{N_u(f,M)\}$ beyond dimension one.

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Sergio ALBEVERIO (Universitä t Bonn)

Recent developments in stochastic analysis related to quantum fields and hydrodynamics

Some new developments are presented, involving construction and study of interactions in quantum fields using stochastic integrals. Related problems in hydrodynamics (with deterministic or stochastic forces) are also discussed.

Andrew D. BARBOUR (Universität Zürich)

Asymptotic behaviour of a metapopulation model

A simple metapopulation model consists of N identical patches. Within each patch, the population evolves as a logistic birth and death process, but it is also subject to random catastrophes. Recolonization occurs because of random migration between patches. For large N, the stochastic evolution can be approximated by the solution of an infinite system of ODE's; we are interested in its equilibria. One such is that all patches are empty, but we show that there may be another, corresponding to a stochastic 'quasi-equilibrium', which then attracts most solutions of the system. Our method of proof involves stochastic comparison arguments, based on couplings of the birth and death processes describing individual patch dynamics. This is joint work with Andrea Pugliese.

Jean BERTOIN (Université Pierre et Marie Curie)

Some aspects of additive coalescents

The additive coalescence is a simple Markovian model for random aggregation that arises for instance in the study of droplet formation in clouds [12, 10], gravitational clustering in the universe [16], phase transition for parking [8],... As long as only finitely many clusters are involved, it can be described as follows. A typical configuration is a finite sequence $x_1 \geqslant \cdots \geqslant x_n \geqslant 0$ with $\sum_1^n x_i = 1$, that may be thought of as the ranked sequence of masses of clusters in a universe with unit total mass. Each pair of clusters, say with masses x and y, merges as a single cluster with mass $x+y$ at rate $K(x,y) = x+y$, independently of the other pairs.
Dealing with a finite number of clusters may be useful to give a simple description of the dynamics, but is a rather unnatural restriction in practice. In fact, one often should like to work with the infinite simplex

$$\mathcal{S}^{\downarrow} = \left\{ x = (x_1,x_2,\dots) : x_i \geqslant 0 \text{ and } \sum_{i=1}^{\infty} x_i = 1 \right\},$$

endowed with the uniform distance. In this direction, Evans and Pitman [11] have shown that the semigroup of the additive coalescence enjoys the Feller property on $\mathcal{S}^{\downarrow}$. Approximating a general configuration $x \in \mathcal{S}^{\downarrow}$ by configurations with a finite number of clusters then enables us to view the additive coalescence as a Markovian evolution on $\mathcal{S}^{\downarrow}$.
In this talk, we shall survey recent developments in this area. In particular, we shall first point at key relations in the finite setting with random trees and certain elementary bridges with exchangeable increments. Then we shall discuss asymptotics when the coalescent starts with a finite but large number of small clusters, and make the connection with Aldous' continuum random tree on the one hand, and the Brownian excursion on the other hand.
Next, we shall turn our attention to a family of non-linear differential equations introduced by Smoluchowski

[19] to model the evolution in the hydrodynamic limit of a particle system in which particles coagulate pairwise as time passes. This bears natural connections with the stochastic coalescence; we refer to the survey by Aldous [1] for detailed explanations, physical motivations, references,... We shall point at a remarkable connection between the so-called eternal solutions to this equation and a certain family of Lévy processes with no positive jumps.

Finally, we shall show that the additive coalescence also arises naturally in the study of sticky particle systems. More precisely, sticky particle systems evolve according to the dynamics of completely inelastic shocks with conservation of mass and momentum, which are also known as the dynamics of ballistic aggregation. This means that the velocity of particles only changes in case of collision, and more precisely, when a shock between (clusters of) particles occurs, a heavier cluster merges at the location of the collision, with mass and momentum given by the sum of the masses and momenta of the clusters involved into the shock. This has been proposed as a model for the formation of large scale structures in the universe; see the survey article [18]. We now have two quite different dynamics for clustering: the ballistic aggregation which is deterministic, and the additive coalescence which is random and may appear much more elementary, as it does not take into account significant physical parameters such as distances between clusters and the relative velocities. Nonetheless, there is a striking connection between the two when randomness is introduced in the deterministic model at the level initial velocities.

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Hakim BESSAIH (Università Pisa)

A mean field result, with application to $3d$ vortex filaments

The importance of thin vortex structures in $3d$ turbulence has been discussed intensively in the last ten years. Some mathematical models of vortex filaments, based on stochastic processes, have been proposed by A. Chorin, G. Gallavotti, A. Majda, F. Flandoli. The importance of these models for the statistics of turbulence or for the understanding of $3d$ Euler equations is under investigation. The hope is based on the numerical results and related physical ideas, and on various results proved in the theory of point vortices in $2d$. The limit properties (propagation of chaos and mean field) of a collection of many interacting vortices has been investigated by P. L. Lions and A. Majda for a particular model of "nearly parallel" vortices.
The aim of our work is to investigate a similar limit for the model introduced by Flandoli. Here the expression for the kinetic energy is not approximated and filaments may fold, so some features are more realistic. However, the filament structures have a fractal cross section (as observed numerically) to eliminate a divergence in the energy.
A mean field result is proved for an abstract model, under two classes of conditions on the rescaling of the energy. Propagation of chaos, variational characterization of the limit Gibbs density h and an equation for h are proved. The general results are applied to a model of $3d$ vortex filaments described by stochastic processes, including Brownian motion and Brownian bridge, other semimartingales, and fractional Brownian motion. This is joint work with Franco Flandoli.

Philippe BLANCHARD (University of Bielefeld)

Knowledge of the exponent in scale-free random graphs does not determine the threshold properties

We will discuss a model of scale free random graphs and it is epidemic threshold properties and show that even under very regular assumptions the exponent of the degree distribution alone does not determine whether one has a threshold or not for the epidemic dynamics. What really matters are structural properties of the random graph spaces like connectivity and degree clustering.

Stefano BONACCORSI (Università di Trento)

On a class of stochastic linear Volterra equations

We consider a class of linear Volterra equations of convolution type perturbed by a noise. If the convolution kernel has the special form of a Riesz fractional integration kernel $t^a/\Gamma(a)$, $a \leqslant 2$, then an explicit solution is provided in terms of a generalized Mittag-Leffler's function. This allows to give precise estimates on the stochastic convolution process defined by the equation.

Michael CRANSTON (University of Rochester)

Lyapunov exponents for the parabolic Anderson model

We prove existence of the Lyapunov exponents for the parabolic Anderson model and derive its asymptotics as diffusivity goes to zero. This is joint work with Thomas Mountford.

Ana Bela CRUZEIRO (Universidade de Lisboa)

An asymptotic estimate for the vertical derivatives of the heat kernel associated to the horizontal Laplacian

We deduce an asymptotic estimate for the heat kernel associated to the horizontal Laplacian which is expressed in terms of curvature of the underlying manifold as an application of an integration by parts formula on the (infinite dimensional) Riemannian path space. This is joint work with Paul Malliavin and S. Taniguchi.

Ian DAVIES (University of Wales)

Stochatic heat and Burgers equations and the intermittence of turbulence

Arnol'd and Thom's beautiful classification of caustics (shockwaves) for Burgers equation suggests a similar one for the wavefronts of the corresponding heat equation. We give here a general theorem for Hamiltonian systems characterizing how the level surfaces of Hamilton's principal function (wavefronts) meet the caustic surface in both the deterministic and stochastic cases. We further show how these results can be applied to the stochastic Burgers equation and discuss the application of these results to turbulence for the Burgers velocity field, specifically the intermittence of turbulence.

Azzouz DERMOUNE (Université de Lille 1)

Stability of a stochastic differential equation with measurable drift

Let $d \geqslant 1$ be an integer, $u_0 : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous bounded map, and B denote a d-dimensional Brownian motion. We are interested in the stability of the differential equation $dX_t = \mathbb{E}[u_0(X_0) \,|\, X_t]\, dt + \nu\, dB_t$. More precisely, let $(X^n)_n$ be a sequence of processes which satisfy $dX_t^n = \mathbb{E}[u_0(X_0^n) \,|\, X_t^n]\, dt + \nu_n\, dB_t$ for each n. If $X_0^n \to X_0$, $\nu_n \to \nu$, does the sequence $(X^n)_n$ converges to a process X which satisfies $dX_t = \mathbb{E}[u_0(X_0) \,|\, X_t] dt + \nu\, dB_t$?
A connection between this problem and the stability of the following gas equations

$$\left\{ \begin{array}{rcl} \dfrac{\partial \rho}{\partial t} + \mathrm{div}(u\rho) & = & \dfrac{\nu^2}{2}\Delta(\rho), \\ \dfrac{\partial (u\rho)}{\partial t} + \mathrm{div}(u \otimes u\rho) & = & \dfrac{\nu^2}{2}\Delta(u\rho), \\ \rho(dx,t) & \to & \rho_0(dx), \\ u(x,t)\rho(dx,t) & \to & u_0(x)\rho_0(dx), \text{ weakly as } t \to 0^+, \end{array} \right.$$

is also established.

Franco FLANDOLI (Università Pisa)

Random currents and probabilistic models of vortex filaments

The language of *currents* seems to be very suitable for the sequel. Let $\mathcal{D}$ be the space of smooth 1-forms in R^3 with compact support (generalizations to manifolds and different dimensions may be done). Its dual space $\mathcal{D}'$ is the space of 1-currents. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space. We call *stochastic current* any linear continuous mapping $\mathcal{T}:\mathcal{D}\to L^0(\Omega,\mathcal{A},\mathbb{P})$ (endowed with the topology of convergence in probability). Mappings defined by stochastic integrals, like

$$\varphi \mapsto \int_0^1 \varphi(X_t)\circ dX_t, \tag{8}$$

are typical examples of stochastic currents; here $(X_t)_{t\in[0,1]}$ is a suitable $3d$-stochastic process on $(\Omega,\mathcal{A},\mathbb{P})$, for instance a continuous semimartingale or a Lyons-Zheng process, and the integration is in the Stratonovich sense (Itô or other integrals can be considered as well, depending on the process).
We say that a stochastic current $\mathcal{T}$ has a *pathwise continuous realization* if there exists a measurable mapping $T:\Omega\to\mathcal{D}'$ such that for every $\varphi\in\mathcal{D}$, we have $(\mathcal{T}\varphi)(\omega)=T(\omega)\varphi$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. The existence of a pathwise continuous realization, in the case of the mapping (8), is an interesting problem of pathwise integration and is essential for the sequel. Such realizations exist for many stochastic integrals by general principles on random distributions (Minlos theorem), but for our applications it is interesting to identify Sobolev spaces $[H^{-s}(\mathbb{R}^3)]^3$ with s as good as possible, such that the realization T takes values in such spaces. A by-product of the theory of rough paths of T. Lyons gives information when $C^{k,\alpha}$ topologies are used over the test functions φ ($k=1$ and any $\alpha>0$ for the case of continuous semimartingales). We need however Sobolev topologies, because of a finite energy requirement described later on. For Brownian semimartingales (namely driven by Brownian motion) and for Lyons-Zheng processes, we prove that the pathwise continuous realization of (8) takes values in $[H^{-s}(\mathbb{R}^3)]^3$ for any $s>\frac{d}{2}$ and we have an argument supporting the conjecture that such exponents are optimal in the case of Brownian motion (there are several other results for Itô integrals, for general semimartingales, for processes with finite p-variation with $p\in(1,2)$, with presumably optimal exponents in the case of fractional Brownian motion using a recent result of Nualart, Rovira and Tindel). As a side remark, of interest for the connections with geometric measure theory, there is a connection between stochastic integrals for finite p-variation with $p\in(1,2)$, and flat currents.
The previous set-up and results are motivated by probabilistic models of *vortex filament* structures observed in $3d$ fluids and first introduced by Chorin with the help of processes on the lattice $\mathbb{Z}^3$. To minimize notations, assume that X is $3d$-Brownian motion. Let $(\rho_t)_{t\in[0,1]}$ be a measure-valued stochastic process in $\mathbb{R}^3$ on $(\Omega,\mathcal{A},\mathbb{P})$, adapted to X. Define the stochastic current

$$\varphi\mapsto\mathcal{T}^{(\rho)}\varphi:=\int_0^1 (C_{\rho_t}\varphi)(X_t)\circ\, dX_t$$

where $(C_{\rho_t}\varphi)(y)=\int\varphi(x+y)\,d\rho_t(x)$. Formally, $\mathcal{T}^{(\rho)}\varphi=\langle\xi,\varphi\rangle$, where

$$\xi(x)=\int_0^1 \rho_t(x-X_t)\,\circ dX_t.$$

If (ρ_t) has finite energy in the mean, namely

$$\int_0^1\left(\int\int\frac{1}{|x-y|}\,d\rho_t(x)\,d\rho_t(y)\right)dt<\infty\quad\mathbb{P}\text{-a.s.}$$

then by the previous methods on pathwise continuous realizations of random currents, we have $\xi\in[H^{-1}(\mathbb{R}^3)]^3$, $\mathbb{P}$-a.s.. We interpret the random distribution ξ as a *vorticity field* of a fluid, concentrated along the curve (X_t),

with a variable cross section (ρ_t). The previous regularity property of ξ implies that it defines a velocity field with *finite kinetic energy*. We call ξ a random vortex filament. It is then possible to introduce *Gibbs measures* over vortex filaments, in analogy with the lattice theory of Chorin.
A number of properties of these Gibbs measures have been obtained or are under study. They include exponential integrability and consequent definition also for negative temperature, a mean field result for a concentration of weak filaments, and some explicit computations of the energy spectrum in suitable limits. The dynamics of the random vortex filaments is also under investigation and some preliminary results are available.

Donald GEMAN (Johns Hopkins University and ENS-Cachan)

Stochastic models for coarse-to-fine search

The semantic interpretation of visual data, including object recognition, is the most ambitious goal of computer vision, especially for natural greyscale scenes and with the speed and precision of human vision. I will talk about the consequences of making computational efficiency the organizing principle for vision, concentrating on a statistical formulation and analysis. Not surprisingly, efficiency leads to multiresolution and hierarchical representations with "hypothesis tests" at varying levels of cost, power (discrimination) and invariance (precision of the interpretation). Two major themes are then the tradeoff between power and invariance, and articulating conditions under which coarse-to-fine processing minimizes mean computation. These ideas will be illustrated by detecting objects in natural images, and related to control mechanisms for selective attention in biological vision. This is joint work with Gilles Blanchard (mathematical results) and François Fleuret (experimental results).

Mihaï GRADINARU (Université de Nancy I)

Itô's formula for fractional Brownian motion

Fractional Brownian motion B^H is not a semimartingale (except for the Hurst index $H = 1/2$). In this talk, we discuss the possibility to obtain Itô's formula for $f(B^H)$, when $H \geqslant 1/4$. The interesting case is $H = 1/4$ when a Bouleau-Yor type identity can be obtained for a 4-covariation process $[f(B^H), B^H, B^H, B^H]$. Proofs are based on the existence of some third order stochastic integrals of $f(B^H)$ against B^H. This is joint work with Francesco Russo and Pierre Vallois.

Peter IMKELLER (Humboldt Universität Berlin)

Stochastic resonance

In an attempt to find a simple qualitative explanation of global glacial cycles, physicists in the beginning of the 1980's started to study systems describing particles moving in multimodal, periodically changing, potential landscapes. Their mathematical description consists in nonlinear differential equations with weak periodic signals of period T, perturbed by white noise with intensity ϵ. The system described by such an equation is in *stochastic resonance* if the intensity ϵ is tuned with the input period T in an optimal way. In the meantime, physics literature abounds with examples of systems showing stochastic resonance with origins in many areas of natural sciences. In my talk, I will report on recent progress in a mathematically rigorous understanding of this phenomenon.

Rémi LÉANDRE (Université de Nancy I)

Classifying spaces, string structure and stochastic Dirac-Ramond operator

The purpose of this talk is twofold. In the first part, we give a stochastic analogue of the algebraic topology proof of Mac Laughlin of the existence of string structure over the based loop space. This requires the understanding of the stochastic cohomology in Chen-Souriau sense of the loop space of the classifying space of a compact Lie group, as well as the functoriality of the stochastic Leray-Serre spectral sequence under the stochastic classifying map of the loop space of the manifold, called the Nemytskii map. In the second part, by using the spin representation of a Kac-Moody group in the sense of Araki and Carey-Ruisjenaars, we establish, when the first Pontryaguin class of the manifold is 0, the existence of a family of stochastic Dirac-Ramond operators which acts over the Hilbert space of section of the spin bundle over the free loop space, associated to a string structure. We establish the existence of stochastic moduli spaces of stochastic connections over the free loop space.

Michel LEDOUX (**Université de Toulouse**)

Large deviations and support theorems for diffusion via rough paths

We use the continuity theorem of T. Lyons for rough paths in the p-variation topology to produce an elementary approach to the large deviation principle and the support theorem for diffusion processes. The proofs reduce to establishing the corresponding results for Brownian motion itself as a rough path in the p-variation topology, $2 < p < 3$, and the technical step is to handle the Lévy area in this respect. Various extensions and applications are discussed. This is joint work with Zhongmin Qian and Tusheng Zhang.

Olivier LÉVÊQUE (**École Polytechnique Fédérale de Lausanne**)

Hyperbolic spde's driven by boundary noises

We study a class of second order linear *hyperbolic* equations driven by Gaussian boundary noise, in the case where the spatial domain is a ball in $\mathbb{R}^d$ and the noise is concentrated on a sphere, or the domain is $\mathbb{R}^d$ and the noise is concentrated on a hyperplane. When the noise is isotropic on the sphere or homogeneous in the hyperplane, we give necessary and sufficient conditions on the covariance of the noise which ensure the existence of a function-valued solution (respectively a random-field solution) to the equation. We examine more precisely the behaviour of the solution near the noisy hyperplane and show that it differs from that of the solution of a parabolic equation driven by boundary noise. This is joint work with Robert C. Dalang.

Sylvie MÉLÉARD (**Université de Paris X**)

Probabilistic interpretation and particle method for $2d$ vortex equations with Neumann's boundary condition

We are interested in proving Monte-Carlo approximations for vortex equations in bounded domains of $\mathbb{R}^2$ with Neumann's condition on the boundary. This work is the first step to justify theorically some numerical algorithms for Navier-Stokes equations in bounded domains with no-slip conditions, as proposed by Chorin or Cottet. We prove that the vortex equation can be interpreted in a probabilistic point of view through a nonlinear reflected process with space-time random births on the boundary of the domain.
Next, we approximate the solution v of this vortex equation by interacting diffusive particles with normal reflecting boundary conditions and space-time random births on the boundary. That allows us to prove the pathwise convergence of an easily simulable particle algorithm. To our knowledge, there was no proof of convergence of such particle algorithms, even for the deterministic ones. This is joint work with Benjamin Joudain.

Annie MILLET (Université de Paris 6)

A discretization scheme for the heat equation perturbed by a space-correlated Gaussian noise

We study the rate of convergence for explicit and implicit discretization schemes of the heat equation on $[0,1]^d$ subject to a random perturbation driven by a Gaussian noise F which is white in space and has a space correlation given by a function f. If $d=1$, T/m (resp. $1/n$) is the time (resp. space) discretization mesh and $f(x)=|x|^{-\alpha}$ for some $\alpha \in [0,1[$, the speed of convergence is equal to $m^{-\frac{2-\alpha}{4}}$ in time and $n^{-\frac{2-\alpha}{2}}$ in space (up to some logarithmic factor); this completes Gyöngy's result in the "limit case" $\alpha = 1$ corresponding to the space-time white noise. We prove similar results for $d \geqslant 2$ when $f(x) = \Pi_{i=1}^d |x_i|^{-\frac{\alpha}{d}}$ for $\alpha \in [0,2[$. This is joint work with Pierre-Luc Morien.

Thomas MOUNTFORD (École Polytechnique Fédérale de Lausanne)

Convergence of exclusion processes

We consider finite range non zero mean excursion processes in one dimension and show that if the initial configuration has a "density" α, then the process must converge to product measure α. This is joint work with Enrique Andjel and Christophe Bahadoran.

Rimas NORVAIŠA (Institute of Mathematics and Informatics Vilnius)

The p-variation calculus and its relation to stochastic analysis

A nonlinear analysis of concrete operators acting on spaces of functions having bounded p-variation is called the p-variation calculus. A typical example of an operator is defined by a solution of an integral equation, and so the calculus for such an operator depends heavily on a construction of an integral with respect to a function. In the talk, we discuss recent results obtained using the Kolmogorov integral with respect to an interval function. The main result is that the solution of a linear Kolmogorov integral equation is a product integral with respect to an interval function, and the induced product integral operator is a uniformly entire mapping between Banach spaces of suitable interval functions of bounded p-variation for some $1 \leqslant p < 2$.
An application of the p-variation calculus to sample functions of a stochastic process gives a straightforward relation to stochastic analysis. In the talk, we also discuss a further more elaborate application of the p-variation calculus to the stochastic analysis of semimartingales. A stochastic process X which is decomposable into a sum of a local martingale M and an adapted càdlàg stochastic process A with locally bounded variation is called a semimartingale. A stochastic process X with the same decomposition property except that A is assumed to have locally bounded p-variation for some $1 \leqslant p < 2$ is called a $(2-\epsilon)$-semimartingale. For example, if B_H is a fractional Brownian motion with Hurst exponent $H \in (0,1)$ and B is a standard Brownian motion, then $B + B_H$ is a $(2-\epsilon)$-semimartingale provided $1/2 < H < 1$. As for semimartingales, a decomposition defining a $(2-\epsilon)$-semimartingale is non-unique in general. Defining an integral with respect to a $(2-\epsilon)$-semimartingale X to be a sum of the stochastic integral with respect to M and the (pathwise) left Young integral with respect to A, the problem of correctness arise due to possible non-uniqueness of such a definition. However, for a class of stochastic processes having the p-variation index equal to 2 almost surely, the indicated integration with respect to a $(2-\epsilon)$-semimartingale is correct, and the resulting indefinite integral is a $(2-\epsilon)$-semimartingale, just like in the usual stochastic analysis of semimartingales. Using our integral with respect to a $(2-\epsilon)$-semimartingale, we define a quadratic covariation between two $(2-\epsilon)$-semimartingales, as in the stochastic analysis of semimartingales. In spite of possible non-uniqueness of a decomposition of a $(2-\epsilon)$-semimartingale,

we prove that the continuous part of the quadratic covariation is defined uniquely. Finally, these results allow us to extend the Itô formula, and show that in the class of all $(2-\epsilon)$-semimartingales, an extended Dolean's exponent gives the unique solution of the linear integral equation with respect to a $(2-\epsilon)$-semimartingale.

Youssef OUKNINE (Faculty of sciences Semlalia, Marrakesh)

Regularity and representation of viscosity solutions of pde's via bsde's

In this paper, we study the regularity of viscosity solution of pde's with Lipschitz coefficients by using the connection with forward backward stochastic differential equations (in short, FBSDE). We also give a probabilistic representation of the generalized gradient (derivative in the sens of distributions) of the viscosity solution to a quasilinear parabolic pde in the spirit of Feyman-Kac formula. In the degenerate case, we use techniques of Bouleau and Hirsch on absolute continuity of probability measure.

Jan ROSINSKI (University of Tennessee)

Boundedness and continuity of infinitely divisible processes

We consider an infinitely divisible process X without Gaussian components indexed by a compact metric space and given as a stochastic integral of a deterministic kernel f with respect to a Poisson random measure. We give strong sufficient conditions for the boundedness and continuity of sample paths of X in terms of a certain Lipschitz-type norm of sections of f and a majorizing measure on the index space. Our present approach generalizes and simplifies previous results on this subject. This is joint work with Michael B. Marcus.

Bernard ROYNETTE (Université de Nancy I)

On independent times and positions for random walks

Let

$$\left(S_n = \sum_{i=1}^{n} X_i; n \geqslant 0\right)$$

be a one-dimensional random walk. We study stopping times T such that $(S_{n\wedge T}; n \geqslant 0)$ is uniformly integrable and T and S_T are independent. This is joint work with Christophe Ackermann and Gérard Lorang.

Barbara RÜDIGER (Universität Bonn)

Stochastic integration with respect to compensated Poisson random measures and the Lévy-Itô decomposition theorem on separable Banach spaces

We give a direct definition of stochastic integration with respect to compensated Poisson random measures on separable Banach spaces. In a paper in collaboration with Sergio Albeverio, this is applied to provide a direct proof of the Lévy-Itô decomposition of a càdlàg process with stationary and independent increments into a jump and Brownian component.

Martha SANZ-SOLÉ (Universitat de Barcelona)

Stochastic wave equation with 3-spatial dimension: existence of density

Consider a stochastic differential equation, spatially homogeneous, whose Green's function is a positive Schwartz distribution with rapid decrease. Extending the definition of Walsh's martingale measure stochastic integral, Dalang in 1999 proved the existence and uniqueness of a real-valued solution, assuming that the noise which drives the equation is white in time and colored in space, with an absolutely continuous correlation measure satisfying some integrability assumptions. This result applies, in particular, to the stochastic wave equation in dimension $d \in \{1,2,3\}$.
We are interested in the study of the existence and smoothness of density for the probability law of the solution $u(t,\underline{x}) = (u(t,x_1),\ldots,u(t,x_m))$ at distinct points $(t,x_1),\ldots,(t,x_m)$, $t > 0$, $x_1,\ldots,x_m \in \mathbb{R}^d$, using Malliavin Calculus. As a first step, we will consider in this talk the existence of density for the random variable $u(t,x)$ solution to the 3-dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_3\right)u(t,x) = \alpha(u(t,x))\dot{F}(t,x) + \beta(u(t,x)),$$
$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0.$$

In order to study the Malliavin differentiability of the random variable $u(t,x)$, we need smoothness on the coefficients and an extension of Dalang's stochastic integration, allowing Hilbert-valued martingale measures as integrators. Then nondegeneracy of the Malliavin variance is proved assuming non degeneracy of α and the following assumption:

$$\text{there exists } \eta \in (0,\tfrac{1}{2}) \text{ such that } \int_{\mathbb{R}^d} \mu(d\xi)\frac{1}{(1+|\xi|^2)^\eta} < \infty,$$
where μ is the spectral measure of the noise.

This hypothesis implies

(i) $C_1 t^{\theta_1} \leqslant \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\hat{\Lambda}(s,\cdot)(\xi)|^2 \leq C_2 t^{\theta_2}$;

(ii) $\int_{\mathbb{R}^d} \mu(d\xi)|\xi||\hat{\Lambda}(s,\cdot)(\xi)|^2 \leq C_3 t^{\theta_3}$,

for suitable positive constants C_i, $i = 1,\ldots,3$, $\theta_1 = 3$, $\theta_2 = 3 - 2\eta$, $\theta_3 = 2 - 2\eta$, where $\hat{\Lambda}$ denotes the Fourier transform of the fundamental solution to the wave equation. All these estimates are crucial in the analysis of the Malliavin variance, which can be done by a comparison procedure with respect to smooth approximations of Λ obtained by convolution.

Richard SOWERS (University of Illinois)

Averaging, glueing conditions and singular perturbations

We discuss some recent work concerning stochastic averaging and Markov processes on stratified spaces. We indicate the connection between glueing conditions at different strata and singular perturbations analyses near certain types of bifurcations.

Michèle THIEULLEN (Université de Paris 6)

Invariance of conditional probabilities: application to geometric Brownian motion

Let us define the reciprocity class of the Wiener measure under a functional Y as the set of probabilities whose conditioning given Y is the same as that of the Wiener measure. We study transformations which preserve this class. As am example we consider the case when Y is the exponential functional of geometric Brownian motion. This is a joint work with Fabrice Baudoin.

Samy TINDEL (**Université de Paris 13**)

Almost-sure Lyapunov exponent of a parabolic spde

We derive a lower bound on the large-time exponential behavior of the solution to a stochastic parabolic partial differential equation on $\mathbb{R}_+ \times \mathbb{R}$ in the case of a spatially homogeneous Gaussian potential that is white-noise in time, and study the relation between the lower bound and the potential's spatial modulus of continuity.

Pierre VALLOIS (**Université de Nancy I**)

Asymptotic estimate for the ruin problem

Let X be a Lévy process, vanishing at 0, and such that $\mathbb{E}[|X_t|] < +\infty$. Then

(1) $$X_t = -ct + \sigma B_t + J_t,$$

where $(B_t\,; t \geqslant 0)$ is a standard Brownian motion, $(J_t; t \geqslant 0)$ is a pure jump process, $B_0 = J_0 = 0$; the two processes $(B_t\,; t \geqslant 0)$ and $(J_t; t \geqslant 0)$ being independent. Let $T_x(X)$ be the hitting time of level x by X: $T_x(X) = \inf\{t \geqslant 0, X_t > x\}$. We adopt the convention $\inf \emptyset = +\infty$.
In insurance risk models, $(-J_t; t \geqslant 0)$ is a compound Poisson process which represents the risk of an insurance compagny, c is the premium rate, x is the initial risk reserve, $(B_t\,; t \geqslant 0)$ is a random perturbation and $x + ct - \sigma B_t - J_t$ is the aggregate claim amount. The question is: when the ruin occurs? The ruin time is obviously defined as

$$T = \inf\{x + ct - \sigma B_t - J_t < 0\} = T_x(X).$$

We know that if $c > E[J_1]$, then X_t goes to $-\infty$ as $t \to +\infty$. Let us defined the ruin probability: $F(x) = P(T_x(X) < +\infty), \quad x > 0$. Let ν be the Lévy measure of X (or J). If ν has some exponential moments, we are able to prove that F admits the following asymptotic development ($x \to +\infty$):

$$F(x) = c_0 e^{-\gamma_0 x} + \sum_{i=1}^{n} \left(c_i e^{-\gamma_i x} + \overline{c}_i e^{-\overline{\gamma}_i x} \right) + o(e^{-ax}),$$

where γ_0 is a real number, and $0 < \gamma_0 < Re(\gamma_1) < \cdots < Re(\gamma_n) < a$.
We generalize an approach developed by Doney and Bertoin & Doney. These authors proved that F decays exponentially to 0, as $x \to +\infty$ (*i.e.* the development of $F(x)$ is restricted to the first term). This is joint work with Bernard Roynette and Agnès Volpi.

Some references

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[2] Doney R. A. Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.*, 44(2):566–576, 1991.

Keith WORSLEY (**McGill University**)

The geometry of random fields in astrophysics and brain mapping

The geometry in the title is not the geometry of lines and angles but the geometry of topology, shape and knots. For example, galaxies are not distributed randomly in the universe, but they tend to form clusters, or sometimes strings, or even sheets of high galaxy density. How can this be handled statistically? Here, concepts from differential topology, integral geometry and random fields are important. The Euler characteristic (EC) of the set of high density regions has been used to measure the topology of such shapes; it counts the number of connected components of the set, minus the number of 'holes,' plus the number of 'hollows.' Despite its complex definition, the exact expectation of the EC can be found for some simple models, so that observed EC can be compared with expected EC to check the model. A similar problem arises in human brain mapping, where the EC is used to detect local increases in blood flow (positron emission tomography) or oxygenation (functional magnetic resonance imaging) due to an external stimulus. This allows researchers to find the regions of the brain that are 'activated' by the stimulus. Recent results of David Siegmund, Jiayang Sun, Satoshi Kuriki, Akemichi Takemura and Jonathan Taylor link this to the volume of tubes and Steiner's formula. Extending these ideas to manifolds, we can detect changes in brain shape via structure masking, surface extraction, and $3d$ deformation fields. Finally, in organic physical chemistry, vortices in excitable media are low density sets that form random closed loops. These are stable if they are linked or knotted, so the question is whether this can actually happen.

Mario WSCHEBOR (**Universidad de la Republica**)

Condition numbers in optimization problems and extrema of random fields

Let A be a Gaussian $n \times m$ random matrix. We consider the feasibility problem of the existence of solutions of the conic system of inequalities $Ax < 0$. A measure for the difficulty of this problem si given by its "Condition Number" $C(A)$ which plays a relevant role in Numerical Analysis and Complexity Theory. The distribution of $C(A)$ is related to the classical unsolved problem in Geometric Probability of computing the probability of covering the $(m-1)$-dimensional sphere with n randomly placed caps having the same angular radius.
The main part of the talk will consist in describing a recent joint result with F. Cucker that permits to give good bounds for the moments of the random variable $\ln[C(A)]$ as functions of n and m. For example, in the relevant case $n \gg m$, one obtains $\mathbb{E}\ln[C(A)] = \max\{\ln m, \ln\ln n\} + O(1)$. The methods are based on the study of the local minima of certain irregular random fields.

Martina ZÄHLE (**Universität Jena**)

Stochastic integrals for fractional random fields and applications to spde's

(Stochastic) forward integrals for fractal random processes as studied in [1] and [2] are extended to random fields. A main tool are partial and mixed fractional integrals and derivatives in $\mathbb{R}^n$. This approach can be applied to spde with fractal noise, in particular, to the semilinear parabolic case. The noises are formal derivatives of random fields of fractional smoothness of order $\geqslant 1/2$, *e.g.* of multifractal Brownian fields or sheets with Hurst exponents $\geqslant 1/2$. The noise coefficients may depend on the unknown field.

Some references

[1] M. Zähle (1998). Integration with respect to fractal functions and stochastic calculus I, *Probab. Th. Relat. Fields* **111**, 333-374.

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Lorenzo ZAMBOTTI (Università Pisa)

Occupation densities for solutions of spde's with reflection

A family of occupation densities for the solution of an spde with reflection is constructed. Several limit formulae are obtained for the reflecting measure, which turns out to be a renormalized local time at 0.

Jean-Claude ZAMBRINI (Universidade de Lisboa)

Stochastic deformation of contact geometry

Contact geometry is a kind of odd-dimensional counterpart of symplectic geometry, whose origin can be traced back to the work of Lie and Klein on the geometrization of ordinary differential equations. We will describe the first steps toward a deformation of contact geometry, making it compatible with stochastic differential equations, and explain why this point of view allows to reinterpret most of the familiar symmetry relations between stochastic processes in an unified way.

Tusheng ZHANG (University of Manchester)

Local times of fractional Brownian sheet

We obtained the existence and continuity of the local time of a fractional Brownian sheet, which extends the classical result for Brownian sheet.

Weian ZHENG (University of California at Irvine)

Rate of convergence in homogenization of parabolic PDEs

We consider the solutions to

$$\frac{\partial u^{(n)}}{\partial t} = a^{(n)}(x)\Delta u^{(n)}$$

where $\{a^{(n)}(x)\}_{n=1,2,\dots}$ are random fields satisfying a "well-mixing"condition (which is different to the usual "strong mixing"condition). We estimate in this paper the rate of convergence of $u^{(n)}$ to the solution of a Laplace equation. Since our equation is of simple form, we get quite a strong result, which covers the previous homogenization results obtained on this equation.

ABSTRACTS

of the

Minisymposium on Stochastic Methods in Financial Models

Jean-Pierre AUBIN (**Université de Paris-Dauphine**)

Dynamic management of portfolios under stochastic and tychastic uncertainty

Thanks to the equivalence formulas between Itô and Stratonovitch stochastic integrals and to the Strook and Varadhan "Support Theorem" and under convenient regularity assumptions, stochastic viability problems are equivalent to invariance problems for control systems, as it has been singled out by Doss in 1977 for instance. It is in this framework of invariance under control systems that problems of stochastic viability in mathematical finance are studied. The Invariance Theorem for control systems characterizes invariance through first-order tangential and/or normal conditions, whereas the stochastic invariance theorem characterizes invariance under second-order tangential conditions. Doss's Theorem states that these *first-order normal conditions are equivalent to second-order normal conditions* that we expect for invariance under stochastic differential equations for smooth subsets.
We extend this result to any subset by defining in an adequate way the concept of contingent curvature of a set and contingent epi-Hessian of a function, related to the contingent curvature of its epigraph. This allows us to go one step further by characterizing functions the epigraphs of which are invariant under systems of stochastic differential equations. We shall show that they are solutions to either a system of first-order Hamilton-Jacobi equations or to an equivalent system of second-order Hamilton-Jacobi equations. This is joint work with Halim Doss.

Vlad BALLY (**Université du Mans**)

A quantization algorithm for pricing and hedging American options

The quantization algorithm permits to solve numericaly certain non-linear problems, namely pricing american options. In a series of papers in collaboration with G. Pages and J. Printemps, we presented a basic variant of this algorithm. In a recent work (which is the subject of my talk), we improved the algorithm in the sense replacing a zero order scheme with a scheme of order one (i.e. piecewise constant interpolation by piecewise linear interpolation). In order to do this, we use Malliavin's integration by parts formula. As a by product, we compute the strategy.

Olé E. BARNDORFF-NIELSEN (**University of Aarhus**)

Power variation: asymptotic distribution theory and financial economics

Power variation is the generalisation of quadratic variation corresponding to considering r-th powers ($r > 0$) of absolute increments. Asymptotic distribution results are derived for a general class of stochastic volatility (SV) models, and some of the results are extended to the setting of multivariate SV models. The theory is illustrated by simulations and by applications to some high-frequency exchange-rate series.

Giovanni BARONE-ADESI (**Università della Svizzera italiana**)

Electricity derivatives

Electricity markets are becoming a popular field of research amongst academics because of the lack of both appropriate models for describing electricity price behavior and for pricing derivatives instruments. Models for price dynamics must take into account seasonalities and spiky behavior of jumps, which seems hard to model by standard jump processes. Without good models for electricity price dynamics, it is difficult to think about good models for futures, forward, swap or option pricing. As a result, very often *practitioners* are reduced

to the employment of models based on the cost-and-carry for their needs. Unfortunately, there are several reasons which discourage the use of this kind of models. The most important arguments against them are related to the physical and temporal constraints of electricity as a non-storable commodity. Non-storability implies that arbitrage arguments cannot be used in defining a pricing model for electricity as the underlying of a derivative contract. Secondly, electricity transportation is based on the availability of line connections and line connections can be damaged by rare and extreme events. Probably this kind of risk should be included in a good pricing model.
In this paper, we do not try to solve all of the above problems, but we simply propose an algorithm for pricing derivative instruments which is based on some intuitions easily observable from market data. The main idea is to include in a discrete time model for electricity price dynamics features revealed by simple time series analysis and to use jointly binomial and Monte-Carlo methods for pricing derivatives like options. Moreover, to reduce the time spent in simulations, we show how the inclusion of a sufficient condition for early option exercise allows us to gain efficiency, without introducing substantial bias in pricing. This is joint work with Andrea Gigli.

Patrick CHERIDITO (**ETH-Zürich**)

Mixed fractional Brownian motion

We show that the sum of a Brownian motion and a non-trivial multiple of an independent fractional Brownian motion with Hurst parameter $H \in (0{,}1]$ is not a semimartingale if $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},\frac{3}{4}]$, that it is equivalent to a multiple of Brownian motion if $H = \frac{1}{2}$ and equivalent to Brownian motion if $H \in (\frac{3}{4},1]$. As an application, we discuss the price of a European call option on an asset driven by a linear combination of a Brownian motion and an independent fractional Brownian motion.

Rama CONT (**École polytechnique**)

Pricing and hedging options with a mis-specified model

The traditional approach to option pricing and hedging either assumes that the underlying follows a known stochastic process, ignoring the "model risk" or specification error related to picking the wrong model. In practice misspecified models are used but their paramteres are recalibrated to market prices in order to correct for the misspecification bias. We propose a framework to analyze how this procedure affects pricing and hedging and study some examples. We show in particular that the stability of the calibration method used strongly affects the misspecification risk.

Ernst EBERLEIN (**University of Freiburg**)

Modelling of Lévy term structures

In standard mathematical finance, Brownian motion plays the dominating role as the driving process for modelling risk factors such as equity prices, indices, interest rates, or volatilities. In order to achieve a better fit to empirical data, it is preferable to replace Brownian motion by a Lévy process. In particular, generalized hyperbolic Lévy motions are processes which allow an almost perfect fit to financial data. We shall first give a brief overview of applications of this approach to key areas in finance.
A new approach to credit risk based on the term structure methodology of Heath, Jarrow, and Morton [7] was introduced by Bielecki and Rutkowski [1], [2]. The Bielecki-Rutkowski model takes the information on rating migration and on credit spreads into account and yields an arbitrage-free model of defaultable bonds. Motivated by this approach, we develop an intensity-based credit risk framework for term structure models

driven by Lévy processes.
We begin with the default free instantaneous forward rate $f(t,T)$ given by

$$df(t,T) = \partial_2 A(t,T)\, dt - \partial_2 \Sigma(t,T)^\top \, dL_t,$$

where $A(t,T)$ and $\Sigma(t,T)$ are processes satisfying certain smoothness conditions and (L_t) is a d-dimensional Lévy process which, in the canonical decomposition, can be written as

$$L_t = bt + cW_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds,dx).$$

The corresponding price of a default-free bond is then given by

$$\begin{aligned} B(t,T) \quad &= \quad B(0,T)\exp\left(\int_0^t (r(s) - A(s,T))\; ds + \int_0^t \Sigma(s,T)^\top c\, dW_s \right.\\ &\qquad \left. + \int_0^t \int_{\mathbb{R}^d} \Sigma(s,T)^\top x\,(\mu^L - \nu^L)(ds,dx) \right). \end{aligned}$$

Now assume that an internal rating system $\mathcal{K} = \{1, \dots ,K\}$ is given. Class 1 corresponds to the best possible rating following default-freeness —which is denoted AAA in the Standard & Poor's rating— class K corresponds to default. The instantaneous forward rate for class $i \in \{1, \dots ,K-1\}$ is assumed to satisfy the equation

$$dg_i(t,T) = \partial_2 A_i(t,T)\, dt - \partial_2 \Sigma_i(t,T)^\top \, dL_t^{(i)},$$

where $(L_t^{(i)})$ is given by $L_t^{(i)} = b_i t + c_i W_t + \int_0^t \int_{\mathbb{R}^d} p_i x\,(\mu^L - \nu^L)(ds,dx)$. The dynamics of the conditional bond price based on the forward rate g_i can then be derived in the form

$$\begin{aligned} dD_i(t,T) \quad &= \quad D_i(t-,T)\left((a_i(t,T) + g_i(t,t))\; dt + \int_{\mathbb{R}^d} \Sigma_i(t,T)^\top p_i x\, \left(\mu^L - \nu^L\right)(dt,dx) \right.\\ &\qquad \left. + \Sigma_i(t,T)^\top c_i\, dW_t + \int_{\mathbb{R}^d} \left(e^{\Sigma_i(t,T)^\top p_i x} - 1 - \Sigma_i(t,T)^\top p_i x \right) \mu^L(dt,dx) \right), \end{aligned}$$

where $a_i(t,T) = \frac{1}{2}\left|\Sigma_i(t,T)^\top c_i\right|^2 - A_i(t,T)$. The credit migration process is modelled by a conditional Markov process C on the space of rating classes $\mathcal{K}$. We define $H_i(t) = \mathbb{1}_{\{s \geqslant 0 | C_s = i\}}(t)$ and write for $i \neq j$, $H_{ij}(t)$ for the number of transitions from rating i to rating j in the time interval $[0,t]$. Then the time t price $D_C(t,T)$ of a defaultable bond maturing at time T, which is currently rated at C_t, equals

$$D_C(t,T) \quad = \quad B(t,T) \sum_{i=1}^{K-1} \left(H_i(t) \exp\left(- \int_t^T \gamma_i(t,u)\, du \right) + \delta_i H_{i,K}(t) \right)$$

where $\gamma_i(t,u) = g_i(t,u) - f(t,u)$ is the i-th forward credit spread and $\delta_i \in [0,1)$ the corresponding recovery rate which is assumed to be constant. This defaultable bond price can also be expressed in terms of a risk-neutral valuation formula, *i.e.* as an expectation with respect to a martingale measure. Further extensions of the model to include reorganization of firms and multiple defaults are considered. This is joint work with Fehmi Özkan.

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Jean-Pierre FOUQUE (**North Carolina State University**)

Pricing volatility time scales

In the framework of stochastic volatility models we present evidence for the presence of a short time-scale, which we model as a mean-reversion time of the process driving the volatility. We show that this time-scale persists under risk-neutral measures and has to be dealt with when pricing or hedging derivatives. Singular perturbation techniques are very efficient in capturing the evolution of the skew through a parsimonious set of parameters, that is model independent and easy to calibrate. Then we will show how option maturity cycles can be incorporated in the dynamics of the stock price, and, doing so, how the fit to the skew is greatly improved. In the last part of the talk, we will show that this perturbative approach leads to an efficient variance reduction technique in Monte Carlo simulation methods for pricing derivatives of various types. This is a joint work with George Papanicolaou, Ronnie Sircar and Knut Solna, based on the book "Derivatives in Financial Markets with Stochastic Volatility" (Cambridge University Press, 2000)

Hélyette GEMAN (**ESSEC and Université Paris Dauphine**)

Pricing and hedging in incomplete markets

Complete markets deliver replication and unique pricing at cost of replication. In practice however, incomplete markets are the norm: the well-documented existence of jumps and stochastic volatility renders equity markets incomplete with just dynamic trading in the underlying asset; some markets such as credit, electricity, weather are incomplete by nature. Rather than using expected utility or super-replication arguments, we generalize the no-arbitrage assumption by the one of no strictly acceptable opportunity and rederive in this framework the two fundamental theorems of asset pricing. When markets are not acceptably complete, we show that the bid-ask spread per unit of contingent claim bought or sold is an increasing function of the size of the position.

Fausto GOZZI (**Università Pisa**)

Degenerate parabolic pde's arising in pricing multiasset European contingent claims: regularity propeties and Galerkin finite element approximation

We consider a family of partial differential equations (pde) that includes the ones satisfied by the no-arbitrage price of European contingent claims in a complete market with several risky assets. The main aim is twofold: to find regularity properties for the solution of such pde's; and to use such regularity properties as a basis for proving convergence of numerical approximation (via the Galerkin finite element method) to the solution. In particular, we concentrate our attention on derivative contracts characterized by discontinuities in the terminal conditions. This is joint work with Roberto Monte, Simona Sanfelici and Vincenzo Vespri.

Max-Olivier HONGLER (École Polytechnique Fédérale de Lausanne)

The right time to sell a stock whose price is driven by Markovian noise

Consider a stock whose price is the solution of the same equation as a geometric Brownian motion, but with the white noise replaced by a two-state continuous-time Markov chain, with an exponential (λ) holding time in each state. As in the standard problem considered in B. Oksendal's classical book, the sale is subject to a fixed cost and the objective is to maximize the expected discounted sale price. We show that the structure of the solution varies according to the possible relationships between the various parameters of the problem and leads to four distinct cases. The solution is given explicitly, though some quantities are characterized as roots of transcendent equations. The solution requires no stochastic calculus but uses many elements from the general theory of optimal stopping for Markov processes, and provides insight into the so-called "principle of smooth fit," which is satisfied in only two of the cases. In the limit $\lambda \to \infty$, where the Markovian noise converges to white noise, the solution converges to that of the standard problem. This is joint work with Robert C. Dalang.

Claudia KLÜPPELBERG (Munich University of Technology)

Optimal portfolios with bounded capital-at-risk

We consider a Black-Scholes type of market consisting in the simplest case of one *riskless bond* and one *risky stock*. Their price processes P_0 and P evolve according to the equations

$$\begin{aligned} P_0(t) &= e^{rt}, \quad t \geqslant 0, \\ P(t) &= p\exp(bt + L(t)), \quad t \geqslant 0, \end{aligned}$$

where $r \in \mathbb{R}$ is the riskless rate, $p > 0$, $b \in \mathbb{R}$. The fluctuations of the risky asset are modelled by the Lévy process $(L(t))_{t\geqslant 0}$ with Lévy-Khintchine representation

$$\mathbb{E}\exp(isL(t)) = \exp(t\Psi(s)), \quad t \geqslant 0,$$

where

$$\Psi(s) = ias - \beta^2\frac{s^2}{2} + \int_{-\infty}^{\infty}\left(e^{isx} - 1 - isx\mathbb{1}_{\{|x|\leqslant 1\}}\right)\nu(dx).$$

(a,β,ν) is called the characteristic triplet. The quantity $\beta^2 \geqslant 0$ denotes the variance of the Wiener component and the Lévy measure ν satisfies $\nu(\{0\}) = 0$ and $\int(x^2 \wedge 1)\,\nu(dx) < \infty$. It indicates that a jump of size x occurs at rate $\nu(dx)$.

Let $\pi(t) = \pi \in [0,1]$ for $t \in [0,T]$ (T denotes a fixed planning horizon) be the portfolio; *i.e.* the fraction of wealth, which is invested in the risky asset. Denoting X^π the *wealth process*, it follows the dynamic

$$X^\pi(t) = x\exp((r + \pi(b-r))t)\mathcal{E}(\pi\widehat{L}(t)), \quad t \geqslant 0,$$

where $\mathcal{E}(\widehat{L}) = \exp L$, *i.e.* $\ln\mathcal{E}(\pi\widehat{L})$ is again a Lévy process.

Whereas the classical mean-variance criterion of portfolio optimization consists in maximizing the expected terminal wealth under a constraint on the variance as a risk measure, we use the *mean-Capital-at-Risk criterion*, where the Capital-at-Risk is the excess risk above the riskless investment. More precisely,

$$\mathrm{CaR}(x,\pi,T) = xe^{rT} - \mathrm{VaR}(x,\pi,T) = xe^{rT}\left(1 - z_\alpha e^{\pi(b-r)T}\right)$$

where z_α is the α-quantile of $\mathcal{E}(\pi\widehat{L}(T))$; *i.e.*

$$z_\alpha = \inf\{z \in \mathbb{R} : P(\mathcal{E}(\pi\widehat{L}(T)) \leqslant z) \geqslant \alpha\}.$$

The optimization problem is then

$$\max_{\pi\in[0,1]} \mathbb{E}[X^\pi(T)] \quad \text{subject to} \quad \mathrm{CaR}(x,\pi,T) \leqslant C.$$

In general the CaR or VaR cannot be calculated explicitly. We invoke an idea of Asmussen and Rosinski [1], which has been used for the simulation of Lévy processes:

$$L(t) \quad \approx \quad \mu(\varepsilon)t + (\beta^2 + \sigma^2(\varepsilon))^{\frac{1}{2}}\widetilde{W}(t) + N^\varepsilon(t), \quad t \geqslant 0,$$

where

$$\begin{aligned} \sigma^2(\varepsilon) &= \int_{|x|<\varepsilon} x^2\nu(dx), \\ \mu(\varepsilon) &= a - \int_{\varepsilon\leqslant|x|\leqslant 1} x\,\nu(dx), \\ N^\varepsilon(t) &= \sum_{s\leqslant t} \Delta L(s)\mathbb{1}_{\{|\Delta L(s)|\geqslant\varepsilon\}}. \end{aligned}$$

The approximation is a consequence of a functional central limit theorem which holds under certain conditions on the small jumps. It means that small jumps ($<\varepsilon$) are approximated by Brownian motion, large ones ($\geqslant\varepsilon$) constitute a compound Poisson process N^ε. We extend this result to an approximation of the VaR, getting for the α-quantile of $\mathcal{E}(\pi\widehat{L}(T))$

$$z_\alpha \approx z_\alpha^\varepsilon(\pi) = \inf\{z \in \mathbb{R} : P(\gamma_\pi^\varepsilon T + \pi(\beta^2 + \sigma_L^2(\varepsilon))^{1/2}W(T) + M_\pi^\varepsilon(T) \leqslant \ln z) \geqslant \alpha\},$$

where we have used the approximation (M_π^ε is a compound Poisson process)

$$\begin{aligned} \ln\mathcal{E}(\pi\widehat{L}(t)) &\approx \gamma_\pi^\varepsilon t + \pi(\beta^2 + \sigma_L^2(\varepsilon))^{1/2}W(t) + M_\pi^\varepsilon(t), \\ \gamma_\pi^\varepsilon &= \pi(\mu_\varepsilon + \frac{1}{2}\beta^2(1-\pi)), \\ M_\pi^\varepsilon(t) &= \sum_{s\leqslant t} \ln(1 + \pi(e^{\Delta L(s)1(|\Delta L(s)|>\varepsilon)} - 1)). \end{aligned}$$

From this we conclude

$$\begin{aligned} \mathrm{VaR}(x,\pi,T) &\approx x z_\alpha^\varepsilon(\pi)\exp((\pi(b-r)+r)T) \\ \mathrm{CaR}(x,\pi,T) &\approx x e^{rT}\left(1 - z_\alpha^\varepsilon(\pi)e^{\pi(b-r)T}\right) \end{aligned}$$

This result applies to various examples, which have been suggested as price processes.

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Arturo KOHATSU-HIGA (**Universitat Pompeu Fabra**)

An insider with progressive enlargement of filtrations

We consider an insider that has information of a future event which is deformed by some independent noise whose variance decreases to zero as the resolution time approaches. This problem provides an example of progressive enlargement of filtrations. We prove that the utility of the insider is finite and that there is no arbitrage if the rate at which the variance of the blurring noise goes to zero is slow enough.

Dilip MADAN (**University of Maryland**)

Stochastic volatility for Lévy procesess

It has been clear that the standard option pricing model of Black-Scholes and Merton has been inconsistent with options data for at least a decade. The empirical performance of a potential successor to BMS is usually measured in terms of hedging and/or pricing performance. To improve on the pricing and hedging performance of the BMS model, the majority of the research has been directed towards modifying the continuous time stochastic process followed by the underlying asset. In particular, asset returns have been modeled as diffusions with stochastic volatility (eg. Hull and White or Heston), as jump-diffusions (eg. Merton or Kou), or both (eg. Bates or Duffie, Pan, and Singleton). Empirical work on these models has generally supported the need for both stochastic volatility and jumps. Stochastic volatility appears to be needed to explain the variation in strike at longer terms, while jumps are needed to explain the variation in strike at shorter terms. On the theoretical side, arguments have been proposed by Geman, Madan, and Yor which suggest that price processes for financial assets must have a jump component, while they need not have a diffusion component. Their argument rests on recognizing that all price processes of interest may be regarded as Brownian motion subordinated to a random clock. This clock may be regarded as a cumulative measure of economic activity, as conjectured by Clark , and as estimated by Ané and Geman. As time must be increasing, the random clock can be modelled as a pure jump increasing process, or alternatively as a time integral of a positive diffusion process, and thus devoid of a continuous martingale component. If jumps are suppressed, then the clock is locally deterministic, which they rule out a priori. Thus, the required jumps in the clock induce jumps in the price process, while no argument similarly requires that prices have a diffusion component.
The explanation usually given for the use of jump diffusion models is that jumps are needed to capture the large moves that occasionally occur, while diffusions are needed to capture the small moves which occur much more frequently. However, since at least the pioneering work of Mandelbrot on stable processes, it has been recognized that many pure jump models are able to capture both rare large moves and frequent small moves. Motivated by the possibility that price processes could be pure jump, several authors have focussed attention on pure jump models in the Lévy class. Technically, these processes can capture frequent small moves through the use of a Lévy density whose spatial integral is infinite.
There are at least three examples of such pure jump infinite activity Lévy processes. First, we have the normal inverse Gaussian (NIG) model of Barndorff-Nielsen, and its generalization to the generalized hyperbolic class by Eberlein, Keller, and Prause. Second, we have the symmetric variance gamma (VG) model studied by Madan and Seneta and its asymmetric extension studied by Madan and Milne , Madan, Carr, and Chang. Finally, we have the model developed by Carr, Geman, Madan, and Yor (CGMY), which further generalizes the VG model. CGMY study the empirical adequacy of the VG and CGMY models in explaining equity option prices across the strike range. They find that these models can explain the so-called volatility smile, and that the empirical performance of these models is typically not improved by adding a diffusion component for returns. These results raise the disturbing question as to whether diffusion components are needed at all when modeling asset returns.
The empirical success of pure jump Lévy processes is not maintained when one considers the variation of

option prices across maturity. It has been observed in Konikov and Madan that these homogeneous Lévy processes impose strict conditions on the term structure of the risk-neutral variance, skewness, and kurtosis. Specifically, the variance rate is constant over the term, skewness is inversely proportional to the square root of the term, while kurtosis is inversely proportional to the term. In contrast, the data suggests that these risk-neutral moments are often rising with term. Collectively, these considerations suggest that it may be desirable to incorporate a richer behavior across maturity than is implied by homogeneous Lévy processes.
In a parallel development in the literature, it has been observed by several author such as Engle, Bates, Heston, Duan, and Barndorff-Nielsen and Shephard, that volatilities estimated from the time series are usually clustered, which is commonly referred to as volatility persistence. This persistence is inconsistent with homogeneous Lévy processes, and possibly explains the failure of such processes to explain option prices across the maturity dimension.
For these reasons, the objective of this paper is to extend the otherwise fairly successful Lévy process models cited above by incorporating stochastic and mean-reverting volatilities. We take three homogeneous Lévy processes, viz the NIG,VG, and the $CGMY$ models, and generate the desired volatility properties by subordinating them to the time integral of a Cox, Ingersoll, and Ross (CIR) process. The randomness of the CIR process induces stochastic volatility, while the mean reversion in this process induces volatility clustering. We term the resulting processes $NIGSV$, $VGSV$, and $CGMYSV$ in recognition of their synthesis with stochastic volatility. These processes are tractable in that analytical expressions can be derived for their characteristic functions. On employing their exponentials to describe stock prices, European options can be priced via Fourier methods as described in Bakshi and Madan and Duffie, Pan and Singleton. In particular, the current paper applies the fast Fourier transform (FFT) method, which is developed in Carr and Madan.
In constructing risk-neutral price processes from the $NIGSV$,$VGSV$, and $CGMYSV$ processes, two approaches are followed. The approaches differ in terms of the filtration in which the martingale condition is based on. The first approach assumes that investors can only condition trades on the level of the stock price, while the second approach assumes that trades can also be conditioned on the level of the Lévy process and the time on the new clock. Thus, the first approach prohibits arbitrages based only on the stock price, while the second approach further precludes arbitrages based on the level of the driving Lévy process and the new clock. The reason that the two approaches were tried is that one can argue that the stock price is far more observable in practice than either of the variables used to model the stock price process.
To operationalize the first approach, we construct the risk-neutral distribution for the stock price at each future time as the exponential of $NIGSV$,$VGSV$, and $CGMYSV$ processes, normalized to reflect the initial term structure of forward prices. This procedure ensures that spot-forward arbitrage is not possible. We also exclude arbitrages involving calendar spreads of options as these also require knowledge of just the stock price (at the earlier maturity) The class of models generated by excluding price-based arbitrages are termed $NIGSA$,$VGSA$, and $CGMYSA$ respectively. The second approach is operationalized by compensating the pure jump processes $NIGSV$,$VGSV$, and $CGYMSV$ to form martingales. These martingales are then stochastically exponentiated to yield martingale candidates (in the enlarged filtration of the Lévy process and the integrated CIR time change) for forward prices. This class of models is termed $NIGSAM$,$VGSAM$, and $CGMYSAM$ respectively. Characteristic functions for the log of the stock price are formulated analytically in all 6 cases. These characteristic functions are used to generate model option prices numerically, which are then compared with the data.
We note that the $NIGSAM$, $VGSAM$,and $CGMYSAM$ models are martingales with respect to the enlarged filtration, which includes knowledge of the driving Lévy process and knowldege of the subordinator given by the time-integrated CIR process. To the extent that these two processes can not be separately ascertained from a time series of prices, serious issues arise as to the practical relevance of the associated martingale condition. Working with purely discontinuous price processes, Geman, Madan, and Yor provide a precise formulation of conditions under which the two processes can be determined from the time series of underlying asset prices. Even if the two processes can be determined from a time series, it is unlikely that the rich dynamics of the option price matrix can be adequately captured by a martingale which reflects movements in only two processes. Hence, if the market is precluding arbitrage based on a richer filtration than the one generated by the two processes, one is again forced to confront the practical relevance of martingale conditions which are based on filtrations that are essentially unobservable.
The models $NIGSA$, $VGSA$, and $CGMYSA$ take a more conservative approach than the martingale models

$NIGSAM$,$VGSAM$, and $CGMYSAM$. Relying only on the ability to observe stock prices, these models generate stock price processes whose risk-neutral expectation is consistent with the initial term structure of forward prices, but which do not require that these forward price be martingales with respect to the filtration generated by the Lévy process and the subordinator. We find that these more conservative models consistently provide substantially superior empirical performance over the models which prohibit arbitrage based on the richer and perhaps unobservable filtration. Given these results, we take up a deeper study of the properties of these more conservative models . In this regard, we introduce two important new concepts, which we term the martingale marginal property and the Lévy marginal property . We define a process as having the martingale marginal property if it has the same marginal distributions as some martingale process. We further define a process as having the Lévy marginal property, if it has the martingale marginal property and if the martingale is derived from normalizing the exponential of a time inhomogeneous Lévy process. We show first that if the CIR process is started at $zero$, then our conservative processes have this Lévy marginal property. When the starting value is not $zero$, we conjecture that these processes have the martingale marginal properties. Although these questions may be investigated computationally by constructing Lévy densities associated with the characteristic functions of the processes, we pursue a richer understanding of the possibilities by structurally reconstructing the one dimensional distributions in alternative ways. This leads to an attractive representation in the form of an inhomogeneous Lévy process perturbed by a process for conditional abnormal returns that are unconditionally absent and eventually zero. Whether trading strategies may be formulated to exploit this information is an open question.
We report the results of estimating all six models using $S\&P500$ option closing prices for the second Wednesday of each month of the year 2000. For other underliers, we report on just the dominating three models $NIGSA$,$VGSA$, and $CGMYSA$. In the interests of brevity, we provide here a sample of quarterly results. The models are observed to be capable of adequately fitting a wide range of strikes and maturities consistently across the year. A detailed study of the pricing errors shows that absolute errors are higher for out-of-the-money options and for shorter maturities. These results suggest directions for further model improvement, but they could at least partially be due to our experimental design, which minimized absolute errors as opposed to relative errors. The optimal design of a heteroskedasticity adjustment is an open question, to be pursued in future research.
Empirical work on options data suggests that there are very few models capable of explaining option prices across both the strike and maturity dimensions. The pioneering study of Bakshi, Cao, and Chen implicitly demonstrates this point as the authors were forced to partition the data by term and moneyness in order to get adequate pricing quality. The only class of models with comparable effectiveness to the models discussed here appear to be the jump-diffusion models studied by Bates, and by Duffie, Pan, and Singleton. These models also employ jump processes and stochastic volatility, but differ from the current paper in that the jump component has jumps occuring rarely (finite activity), thus requiring the use of a diffusion component to capture the frequent small moves of the underlying. This diffusion component must also have stochastic volatility in order to capture the observed strike variation in price of options with longer terms. This is joint work with peter Carr, Héliette Geman and Marc Yor.

Claude MARTINI (ARTABEL, Orsay)

Characterization of cheapest superstrategies in the presence of model uncertainty

1. Introduction. The purpose of this work is to set a framework for dealing with model uncertainty in mathematical finance, more precisely to handle the pricing of contingent claims in this context. The usual way of modelling the dynamic of the underlying asset of the claim is to specify a probability law such that the set of the equivalent probabilities under which the canonical process is a martingale is not empty. This is a sufficient condition to preclude pure gambling strategies that never fail and at the same time win with a positive probability. In this setting, the question arises to superreplicate the payoff of the claim with probability one at an initial cost as low as possible. This problem has been settled first by El Karoui and Quenez [8] for continuous processes and later by Kramkov [12] for locally bounded processes with jumps. The local boundedness assumption has been removed later by Föllmer and Kabanov in [11].

Now practitioners are faced with the estimation of the model parameters, and up to now there is no consensus about the right model to use. A first way to cope with this model uncertainty is to study the robustness of the prices and hedging strategies computed from a given model in a perturbed environment. The first paper in this direction is [9]. A more ambitious approach is to try to set and solve the superreplication problem in the presence of model uncertainty. This has been undertaken by [1] and [13] by stochastic control techniques, in the case of standard European options with smooth payoffs, by specifying an uncertainty on the volatility (Uncertain Volatility Model (UVM)). Nevertheless, a general framework was missing. A first formulation, which allows the extension of the results of [1] and [13] to merely continuous payoffs, was proposed in [14]. A major difficulty is that one is faced with a family of measures which are in general mutually singular. Even worse, this family is typically non-dominated in the statistical sense.
The purpose of this work is to provide a coherent framework in which one can set and solve the superreplication problem, at least for European contingent claims (including path-dependent ones), which encompasses the case of the UVM model.
This framework, in the continuous time case, is that of the space associated to a regular capacity [10] defined by a set of martingale measures on the canonical space: the canonical process stands for the price process of the underlying of the contingent claim at hand. In the discrete time case, the classical balayage theory provides a full characterization result.
This generalization of the classical superreplication problem is also natural from a theoretical point of view: it can be remarked that this problem depends only on the null sets of the probability under which the problem is specified, so that one can choose without loss of generality any of the equivalent martingale measures. Therefore a proof which does not depend on any particular choice of the martingale measure, with a more analytic flavour, is not surprising.

2. Discrete time case. (Joint work with S. Deparis) We are interested in a discrete time financial market with finite horizon, say N, and we model a stock S via its increments $(x_1,\dots,x_N)$ until N. We suppose that the initial value S_0 is equal to 1. Then the underlying's value at n is $1+x_1+\cdots+x_n$. Possible trajectories belong to a subset E of $\Omega=\left\{x\in\mathbb{R}^N : 1+x_1+\cdots+x_n>0,\ n=0,\dots,N\right\}$ (not necessarily a product set). This means in particular that the spot is always positive. The topology of Ω is induced from the natural one of $\mathbb{R}^N$. In this setting, the following characterization result is an application of the balayage theory [5]:

Theorem 1 *[7] Let E be a compact subset of Ω and suppose there exists a martingale measure on $(E,\mathcal{B}_E)$. Let f be a u.s.c. bounded function on E. Then*

$$\begin{aligned}&\inf\left\{\begin{array}{c} c\,/\,\left(c,(\Delta_n)_{0\leqslant n\leqslant N-1}\right), \\ \forall x\in E, c+\sum_{n=0}^{N-1}\Delta_n x_{n+1}\geqslant f\left(x_1,\dots,x_N\right)\end{array}\right\} \\ =&\sup_{\mu:\text{ martingale measure with support in } E}\mu\left(f\right)\end{aligned}\tag{9}$$

where c and Δ_0 in the infimum are constants and $(\Delta_n)_{1\leqslant n\leqslant N-1}$ are continuous functions of the first n coordinates.

3. Continuous time case. (Joint work with L. Denis) We consider a set of martingale laws $\mathcal{P}$ on the canonical space $\Omega=C([0,T],\mathbb{R})$, which satisfies the following property:
Hypothesis (H): there exists a finite deterministic measure on $[0,T]$, μ and a constant $a\in]0,1[$ such that for each $P\in\mathcal{P}$,

$$a\mu(dt)\ \leqslant d\langle B,B\rangle_t\ \leqslant\mu(dt),$$

$\mathbb{P}$-p.s. and there exist two positive constants C, α such that

$$\forall 0\leqslant s\leqslant t,\ \mu([s,t])\leqslant C\mid t-s\mid^{\alpha}.$$

Notice that this encompasses the UVM case.

Definition 1 *For each $f\in C_b(\Omega)$, we set*

$$c(f)=\sup\{(E_P(\mid f\mid^2))^{1/2},\ P\in\mathcal{P}\}.$$

We denote by $\mathcal{L}^1(c)$ the topological completion of $C_b(\Omega)$ with respect to c. For a process $h \in L^2([0,T],\mu;\mathcal{L}^1(c))$, it can be shown that the stochastic integral

$$t \mapsto \int_0^t h_s \, dB_s$$

exists as a process in $\mathcal{L}^1(c)$.

Definition 2 *We denote by K, the space*

$$K = \{\int_0^T h_s \, dB_s \, ; h \in L^2([0,T],\mu;\mathcal{L}^1(c))\}.$$

Consider $f \in \mathcal{L}^1(c)$, which stands for the payoff of a contingent claim. We want to estimate the quantity

$$\Lambda(f) = \inf\{a \, : \, \exists g \in K, \, a + g \geqslant f \ q.e.\}.$$

defined on the subset Π of $\mathcal{L}^1(c)$ on which the defining set in between the curly braces is not empty. Then we have the following partial characterization result.

Theorem 2 *Assume (H). Let $F : \mathbb{R}^3 \to \mathbb{R}$ a continuous map and $f = F(B_T, \int_0^T B_s, \sup_{t\in[0,T]} B_t)$. Then $\Lambda(f) = \sup\{E_P[f], \, P \in \mathcal{P}''\}$, where $\mathcal{P}''$ is a set of martingale probability laws on Ω which also satisfies (H), with the same a,μ as $\mathcal{P}$.*

The proof relies on a compactification method [2], coupled with an image law argument.

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Antonella MIRA (**Universitá dell'Insubria**)

Bayesian estimate of credit risk via MCMC with delayed rejection

We develop a Bayesian hierarchical logistic regression model to predict the credit risk of companies classified in different sectors. Explanatory variables derived by experted from balance sheets are included. Markov chain Monte Carlo (MCMC) methods are used to estimate the proposed model. In particular we show how the delaying rejection strategy outperforms the standard Metropolis-Hastings algorithm in terms of asymptotic efficiency of the resulting estimates. The advantages of the our model over others proposed in the literature are discussed.
This is joint work with P. TENCONI.

Elisa NICOLATO (**University of Aarhus**)

On multivariate extensions of Ornstein-Uhlenbeck type stochastic volatility models and multi-asset options

Univariate stochastic volatility models based on Lévy driven processes of the Ornstein-Uhlenbeck type have been introduced by Barndorff-Nielsen and Shephard. The univariate BN-S models possess authentic capability of capturing the so-called stylized features of financial time series. In this talk, we propose and analyze a continuous time multivariate extension. In particular, we investigate this class of models from the viewpoint of derivative asset analysis. We discuss topics related to the incompleteness of this type of markets and for structure preserving martingale measures, we derive the price of simple multi-asset options in closed form.

Bernt ØKSENDAL (**University of Oslo**)

White noise calculus for Lévy processes with applications to finance

We present a white noise theory for a given Lévy process $\eta(t)$. The starting points are

(i) the chaos expansion in terms of iterated integrals of the power jump processes of η and

(ii) the chaos expansion in terms of iterated integrals with respect to the associated compensated Poisson random measure.

From this, we define the corresponding spaces of stochastic test functions and stochastic distributions, the white noise processes $\dot{\eta}(t)$,$\dot{\eta}^{(m)}(t)$, and the Wick product $\diamond$. We prove that if $Y(t)$ is an adapted process which is integrable w.r.t. $d\eta(t)$, then

$$\int Y(t)\, d\eta(t) = \int Y(t) \diamond \dot{\eta}(t)\, dt.$$

Then we use this to prove a generalized Clark-Haussmann-Ocone (CHO) theorem for Lévy processes. This CHO theorem is then applied to compute replicating portfolios in financial markets driven by Lévy processes. This is joint work with Giulia Di Nunno and Frank Proske.

Marie-Claude QUENEZ-KAMMERER (**Université de Marne-La-Vallée**)

Optimal portfolio in an incomplete multiple-priors model

We are concerned with the problem of maximization of utility from terminal wealth in an incomplete multiple-priors model. Our approach consists in solving a dual minimization problem over the set of local martingale measures relative to the different priors. We study in detail the case of a logarithmic utility function and the case of a power utility function.

Sergio SCARLATTI (**Università di Chieti-Pesacara**)

Pricing barriers under stochastic volatility: an approximation result

We show that a simple expansion method in powers of the correlation coefficient of the asset and volatility processes produces useful approximations for the prices of barrier options on futures.

Gerhard SCHEUENSTUHL (**Risklab Germany, Hypovereinsbank**)

Options and long-term-investment

In turbulent market situations, hedging activities regularly gain in popularity and investors are more willing to invest in option based investment products. However, if such strategies are implemented over a longer investment horizon, it is not obvious what the long term results of such complex strategies will be and how they affect the wealth situation of the investor in the long run. In this article, we will investigate two classical option based investment strategies, the protected put buying (PPB) strategy and the covered call writing (CCW) strategy with respect to their effects in the long run. We will look beyond the traditional pay-off diagram representation of such strategies and investigate the probability distribution of their long term results. Based on the corresponding distribution parameters, we identify different effects of alternative strategies. With that, we will identify adequate asset allocation recommendations depending on the individual circumstances and risk preferences of the investor.

Uwe SCHMOCK (**Universität Zürich**)

Dealing with dangerous digitals

Options with discontinuous payoffs are generally traded above their theoretical Black-Scholes prices, because of the hedging difficulties created by their large delta and gamma values. A theoretical method for pricing these options is to constrain the hedging portfolio and incorporate this constraint into the pricing, by computing the smallest initial capital which permits super-replication of the option. We develop this idea for exotic options, in which case the pricing problem becomes one of stochastic control. The high cost of exact super-replication coincides with market price quotations for dangerous derivatives such as knock-out barrier options, which are often higher than their risk-neutral expected payoff (theoretical value). This is joint work with Steven Shreve and Uwe Wystup.

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Christophe STRICKER (**Université de Franche-Comté**)

Hedging of contingent claims under transaction costs

We consider a general framework covering models of financial markets with transaction costs in the continuous-time setting. Assuming that the solvency cones are proper and evolve in time continuously, we prove a hedging theorem describing the set of initial endowments allowing to hedge a vector-valued contingent claim by a self-financing portfolio.

Agnès SULEM (**INRIA Rocquencourt**)

Risk Sensitive portfolio optimisation with transaction costs

We develop methods of risk sensitive impulse control theory in order to solve an optimal asset allocation problem with transaction costs and a stochastic interest rate. The optimal trading strategy and the risk-sensitized expected exponential growth rate of the investor's portfolio are characterized in terms of a nonlinear quasi-variational inequality. This problem can then be interpreted as the ergodic Isaac-Hamilton-Jacobi equation associated with a min-max problem. We use a numerical method based on an extended two-stage Howard-Gaubert algorithm and provide numerical results for the case of two assets and one factor that is a Vasicek interest rate. This is joint work with Tomasz R. Bielecki, Jean-Philippe Chancelier and Stanley R. Pliska.