Solution 1. First we compute $T_s$, which is the duration of one bit:

$$T_s = \frac{1}{1 \text{ Mbps}} = 10^{-6} \text{ s.}$$

Now, we can calculate the energy of the signal (i.e. the energy per bit), which is the same for every $j$:

$$E_b = 10^2 T_s.$$ 

The bit error probability is given by $Q\left(\frac{\sqrt{E_b}}{\sigma}\right)$. In our case $\sigma = \sqrt{N_0/2} = 10^{-1}$, thus we need to solve

$$10^{-5} = Q\left(\frac{10^{-3} \times b}{10^{-1}}\right) = Q\left(10^{-2} \times b\right),$$

hence $b = Q^{-1}(10^{-5}) \times 10^2 \approx 426.5$.

Solution 2. 

(a) There are various possibilities to choose an orthogonal basis. One is $\phi_1(t) = \frac{w_0(t)}{\|w_0\|} = \sqrt{\frac{1}{T_s}} w_0(t)$ and $\phi_2(t) = \frac{w_2(t)}{\|w_2\|} = \sqrt{\frac{1}{T_s}} w_2(t)$. Another choice, that we prefer and will be our choice in this solution is

$$\psi_1(t) = \sqrt{\frac{2}{T_s}} \mathbf{1}_{[0,T_s]}(t)$$

$$\psi_2(t) = \sqrt{\frac{2}{T_s}} \mathbf{1}_{[T_s,T_s]}(t).$$

With the latter choice the signal space is

$$w_0 = \sqrt{\frac{T_s}{2}} (1,1)^T$$

$$w_1 = \sqrt{\frac{T_s}{2}} (-1,-1)^T$$

$$w_2 = \sqrt{\frac{T_s}{2}} (1,-1)^T$$

$$w_3 = \sqrt{\frac{T_s}{2}} (-1,1)^T$$

\[\psi_2\]

\[\psi_1\]
(b) $U_0 \in \{\pm 1\}$ and $U_1 \in \{\pm 1\}$ are mapped into

$$U_0 \sqrt{\frac{T_s}{2}} \psi_1(t) + U_1 \sqrt{\frac{T_s}{2}} \psi_2(t).$$

The mapping is shown below:

The mapping is such that neighboring points differ by one bit. This minimizes the bit-error probability since when we make an error chances are that we choose a neighbor of the correct symbol. Notice that we may decode each bit independently. In fact the first bit is decoded to a 1 iff the observation is to the right of the vertical axis and the second bit is 1 iff it is above the horizontal axis. The bit error probability is therefore

$$P_b = Q \left( \sqrt{\frac{T_s/2}{N_0/2}} \right) = Q \left( \sqrt{\frac{T_s}{N_0}} \right).$$

(c) Notice that $\psi_2(t) = \psi_1(t - \frac{T_s}{2})$. Hence one matched filter is enough. The receiver block diagram is:

(d) $E_b = \frac{E_s}{2} = \frac{T_s}{2}$ and the power is $\frac{E_s}{T_s} = 1$.

**Solution 3.**

(a) Using the identity $\cos^2(a) = \frac{1}{2} [1 + \cos(2a)]$, the average energy can be computed as

$$\int_{-\infty}^{\infty} |w_i(t)|^2 \, dt = \frac{2E}{T} \int_0^{T} \cos^2(2\pi(f_c + i\Delta f)t) \, dt$$

$$= \frac{2E}{T} \left[ \frac{t}{2} + \frac{\sin(4\pi(f_c + i\Delta f)t)}{8\pi(f_c + i\Delta f)} \right]_0^T$$

$$= \frac{E}{T} \left[ 1 + \frac{\sin(4\pi \Delta f T)}{4\pi(f_c + i\Delta f)} \right] \approx E.$$

The last approximation follows since $f_c \gg \Delta f$ implies the second term in the square brackets is negligible.
(b) Orthogonality requires
\[ \mathcal{E} \frac{2}{T} \int_0^T \cos(2\pi(f_c + i\Delta f)t) \cos(2\pi(f_c + j\Delta f)t) \, dt = 0, \]
for every \( i \neq j \). Using the trigonometric identity \( \cos(\alpha) \cos(\beta) = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta) \), an equivalent condition is
\[ \mathcal{E} \frac{2}{T} \int_0^T \left[ \cos(2\pi(i - j)\Delta ft) + \cos(2\pi(2f_c + (i + j)\Delta f)t) \right] \, dt = 0. \]

Integrating we obtain
\[ \mathcal{E} \left[ \frac{\sin(2\pi(i - j)\Delta fT)}{2\pi(i - j)\Delta f} + \frac{\sin(2\pi(2f_c + (i + j)\Delta f)T)}{2\pi(2f_c + (i + j)\Delta f)} \right] = 0. \]

As \( f_cT \) is assumed to be an integer, the result can be simplified to
\[ \mathcal{E} \left[ \frac{\sin(2\pi(i - j)\Delta fT)}{2\pi(i - j)\Delta f} + \frac{\sin(2\pi(i + j)\Delta fT)}{2\pi(2f_c + (i + j)\Delta f)} \right] = 0. \]

As \( i \) and \( j \) are integer, this is satisfied for \( i \neq j \) if and only if \( 2\pi\Delta fT \) is an integer multiple of \( \pi \). Hence, we obtain the minimum value of \( \Delta f \) if \( 2\pi\Delta fT = \pi \) which gives \( \Delta f = \frac{1}{T} \). Note that once \( \Delta f \) is an integer multiple of \( \frac{1}{T} \) the approximate equality in (*) will be exact.

(c) Proceeding similarly, we will have orthogonality if and only if
\[ \mathcal{E} \left[ \frac{\sin(2\pi(i - j)\Delta fT + \theta_i - \theta_j)}{2\pi(i - j)\Delta f} + \frac{\sin(2\pi(i + j)\Delta fT + \theta_i + \theta_j)}{2\pi(2f_c + (i + j)\Delta f)} \right] = 0. \]

In this case we see that both parts become zero if and only if \( 2\pi\Delta fT \) is an even multiple of \( \pi \), meaning that the smallest \( \Delta f \) is \( \Delta f = \frac{1}{T} \) which is twice the minimum frequency separation needed in the previous part. Hence, the cost of phase uncertainty is a bandwidth expansion by a factor of 2.

(d) The condition for essential orthogonality is that
\[ \mathcal{E} \left[ \frac{\sin(2\pi(i - j)\Delta fT + \theta_i - \theta_j)}{2\pi(i - j)\Delta f} + \frac{\mathcal{E}}{T} \left[ \frac{\sin(2\pi(2f_c(i + j)\Delta fT) + \theta_i + \theta_j)}{2\pi(2f_c + (i + j)\Delta f)} \right] \right] \]
is small compared to the signal’s energy \( \mathcal{E} \). The first term vanishes if \( \Delta f = \frac{1}{T} \). The second term is very small compared to \( \mathcal{E} \) if \( f_cT \gg 1 \).

(e) We have \( m \) signals separated by \( \Delta f \). The approximate bandwidth is \( m\Delta f \). This means bandwidth \( \frac{2^k}{T} \) without random phase, and bandwidth \( \frac{2^k}{T} \) with random phase. We see that in both cases, \( WT \) is proportional to \( 2^k \), i.e. it grows exponentially with \( k \).

**Solution 4.**
(a) The block diagram is shown below:

\[ R(t) \xrightarrow{w_0(T-t)} t=T \xrightarrow{Y} \hat{H} \]

(b) Given \( A = a \), the distance of signals is \( 2a\sqrt{\mathcal{E}_b} \), hence

\[ P_e(a) = Q \left( a\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right). \]

(c)

\[ P_f = \mathbb{E}[P_e(a)] = \int_{0}^{\infty} Q \left( a\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) 2ae^{-a^2} \, da. \]

We integrate by parts, noting that \( \int 2ae^{-a^2} \, da = -e^{-a^2} : \)

\[ P_f = -Q \left( a\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) e^{-a^2}\bigg|_{0}^{\infty} + \int_{0}^{\infty} Q' \left( a\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) e^{-a^2} \, da. \]

Taking the derivative of an integral with respect to the lower boundary gives the negative of the value of the integrand evaluated at the lower boundary, i.e.,

\[ Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \]

Thus, for the derivative of \( Q \left( a\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) \) with respect to \( a \), we can write

\[ \frac{d}{da} Q \left( a\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} \frac{2\mathcal{E}_b}{N_0} \sqrt{\frac{2\mathcal{E}_b}{N_0}}. \]

Plugging this in, we find

\[ P_f = \frac{1}{2} - \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\mathcal{E}_b}{N_0}} e^{-a^2} \left( \frac{a}{N_0} + 1 \right) \, da, \]

which we now reshape to make it an integral over a Gaussian density, as follows:

\[ P_f = \frac{1}{2} - \sqrt{\frac{2\mathcal{E}_b}{N_0}} \frac{1}{\sqrt{2}} \int_{0}^{\infty} \frac{1}{\sqrt{\left( \frac{a}{N_0} + 1 \right) \left( \frac{a}{N_0} + 1 \right)}} \exp \left( -\frac{a^2}{2 \left( \frac{a}{N_0} + 1 \right)} \right) \, da. \]

Now, it is clear that the integral evaluates to one half (since the integral is only over half of the real line), and we find

\[ P_f = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\mathcal{E}_b/N_0}{1 + \mathcal{E}_b/N_0}} = \frac{1}{2} \left( 1 - \sqrt{\frac{\mathcal{E}_b/N_0}{1 + \mathcal{E}_b/N_0}} \right). \]
(d) Let $\sigma = \frac{1}{\sqrt{2}}$, then

$$m = \mathbb{E}[A] = \int_0^\infty 2a^2 e^{-a^2} da = 2\sqrt{\pi} \int_0^\infty a^2 \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}} da = \sqrt{\pi} \sigma^2 = \frac{\sqrt{\pi}}{2}.$$ 

Thus, using the formula from part (b):

$$P_e(m) = Q\left(\frac{m}{\sigma} \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\frac{\sqrt{\pi} \sigma}{\sqrt{2} \sqrt{\frac{\mathcal{E}_b}{N_0}}}\right).$$

For the given example we get

$$\mathcal{E}_b = \frac{2(Q^{-1}(10^{-5}))^2}{\pi} \approx 10.6 \text{ dB}.$$ 

For the fading we use the result of part (c) to get

$$\mathcal{E}_b = \frac{(1 - 2 \cdot 10^{-5})^2}{1 - (1 - 2 \cdot 10^{-5})^2} \approx 44 \text{ dB}.$$ 

The difference is quite significant! It is clear that this behaviour is fundamentally different from the non-fading case.

**Solution 5.**

(a) In this basis the signal representations are $c_1 = (2, 0, 0, 2)^T$, $c_2 = (0, 2, 2, 0)^T$, $c_3 = (2, 0, 2, 0)^T$, $c_4 = (0, 2, 0, 2)^T$.

(b) The union bound is expressed in terms of the pairwise distances $d_{ij}$ between the signals since

$$P_e(i) \leq \sum_{j \neq i} Q\left(\frac{d_{ij}}{2\sigma}\right)$$

From (a) we observe that $d_{12}^2 = d_{34}^2 = 16$ and $d_{13}^2 = d_{14}^2 = d_{23}^2 = d_{24}^2 = 8$, hence

$$P_e(i) \leq 2Q\left(\frac{2}{\sqrt{N_0}}\right) + Q\left(\frac{2\sqrt{2}}{\sqrt{N_0}}\right)$$

Since $P_e(i)$ does not depend on $i$, it also bounds the average error probability.

(c) The minimum-energy signal set is obtained by subtracting from $\{w_i(t)\}_{i=1}^4$ the average signal $a(t) = \frac{1}{4} \sum_{i=1}^4 w_i(t) = 1_{[0,4]}(t)$. The resulting signals are shown below.
(d) Note that in the new signal set \( \tilde{w}_2(t) = -\tilde{w}_1(t) \) and \( \tilde{w}_4(t) = -\tilde{w}_3(t) \). Furthermore the signals \( \tilde{w}_1(t) \) and \( \tilde{w}_3(t) \) are orthogonal. Thus the new signal space is two-dimensional, and the Gram–Schmidt procedure will produce the orthonormal basis \( \tilde{\psi}_1(t) = \frac{1}{2} \tilde{w}_1(t) \) and \( \tilde{\psi}_2(t) = \frac{1}{2} \tilde{w}_3(t) \).

(e) In the new basis the signal representations are \( \tilde{c}_1 = (2, 0)^T, \tilde{c}_2 = (-2, 0)^T, \tilde{c}_3 = (0, 2)^T, \tilde{c}_4 = (0, -2)^T \). These codewords correspond to those of the 4-QAM constellation (rotated by 45 degrees). The error probability of this set is

\[
P_e = 1 - \left[ 1 - Q\left( \frac{2}{\sqrt{N_0}} \right) \right]^2 = 2Q\left( \frac{2}{\sqrt{N_0}} \right) - Q\left( \frac{2}{\sqrt{N_0}} \right)^2
\]

(f) Since translations of a signal set do not change the probability of error, the error probability of the receiver in (b) is equal to that in (e).

**Solution 6.**

(a) Clearly,

\[ \mathcal{E}_s^C(k) = 2^{2k} - 1. \]

(b)

\[ a = Q^{-1}\left( \frac{10^{-5}}{2} \right) \approx 4.42. \]

(From the suggested approximation we get \( a \approx 4.80 \).)

(c) For comparison, see the following table.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \mathcal{E}_s^P(k) )</th>
<th>( \mathcal{E}_s^C(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19.54</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>97.68</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>1660</td>
<td>255</td>
</tr>
</tbody>
</table>
(d) We see that

$$\frac{E_s^C(k+1)}{E_s^C(k)} = \frac{E_s^P(k+1)}{E_s^P(k)} = \frac{2^{2(k+1)} - 1}{2^{2k} - 1},$$

thus

$$\lim_{k \to \infty} \frac{E_s^C(k+1)}{E_s^C(k)} = \lim_{k \to \infty} \frac{E_s^P(k+1)}{E_s^P(k)} = 4.$$

(e) If we send one bit per symbol, then coding allows us to significantly reduce the required energy per symbol. For every additional bit per symbol we need to multiply $E_s$ by roughly 4 (exactly 4 asymptotically) with or without coding. So as the number of bits per symbol increases, there is essentially a constant gap (in dB) between the energy per symbol required by (uncoded) PAM and that required by the best possible code.

Notice that to keep the error probability at a constant level, we need to increase $E_s/\sigma^2$ exponentially with the number $k$ of bits per symbol. In Example 4.3 in the book we increase it linearly with $k$ (hence the error probability goes to 1).