Handout 21 Advanced Digital Communications
Solutions to Homework 8 Nov. 21, 2016

SOLUTION 1.

$$\begin{aligned}
\overline{P_\mathrm{e}} &= \int_0^\infty Q(\sqrt{\gamma})\frac{1}{\overline{\gamma}}e^{-\gamma/\overline{\gamma}}\,d\gamma \\
&= \int_0^\infty \left(\frac{1}{\sqrt{2\pi}}\int_{\sqrt{\gamma}}^\infty e^{-t^2/2}\,dt\right)\frac{1}{\overline{\gamma}}e^{-\gamma/\overline{\gamma}}\,d\gamma \\
&\overset{(1)}{=} \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\left(\int_0^{t^2}\frac{1}{\overline{\gamma}}e^{-\gamma/\overline{\gamma}}\,d\gamma\right)dt \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\left(1-e^{-t^2/\overline{\gamma}}\right)dt \\
&= \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-t^2/2}e^{-t^2/\overline{\gamma}}\,dt \\
&= \frac{1}{2} - \frac{1}{\sqrt{1+2/\overline{\gamma}}}\underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi/(1+2/\overline{\gamma})}}e^{-\frac{t^2}{2/(1+2/\overline{\gamma})}}\,dt}_{=1/2} \\
&= \frac{1}{2}\left(1-\sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right).
\end{aligned}$$

Step (1) follows from interchanging the order of integration. If you are not familiar with this, look it up, it is a useful trick to know!

SOLUTION 2.

(a) We proceed as in the lecture notes by defining $x := \frac{1}{10}\frac{|h|^2\mathcal{E}}{N_0/2}$ and upper bound the error probability as

$$P_e \leq 3Q(\sqrt{x})$$

Hence the average error probability can be bounded as

$$\bar{P}_e \leq \int_0^\infty 3Q(\sqrt{x})f(x)dx$$

where x a exponentially distributed as

$$f(x) = \begin{cases} \frac{1}{\bar{x}}e^{-x/\bar{x}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

with $\bar{x} = \frac{1}{10}\frac{\mathcal{E}}{N_0/2}$. The same as for the BPSK example in the class, the integration can be calculated explicitly to give

$$\bar{P}_e \leq \frac{3}{2}\left(1-\sqrt{\frac{\bar{x}}{2+\bar{x}}}\right)$$

For large $\bar{x}$ we can use the approximation to get

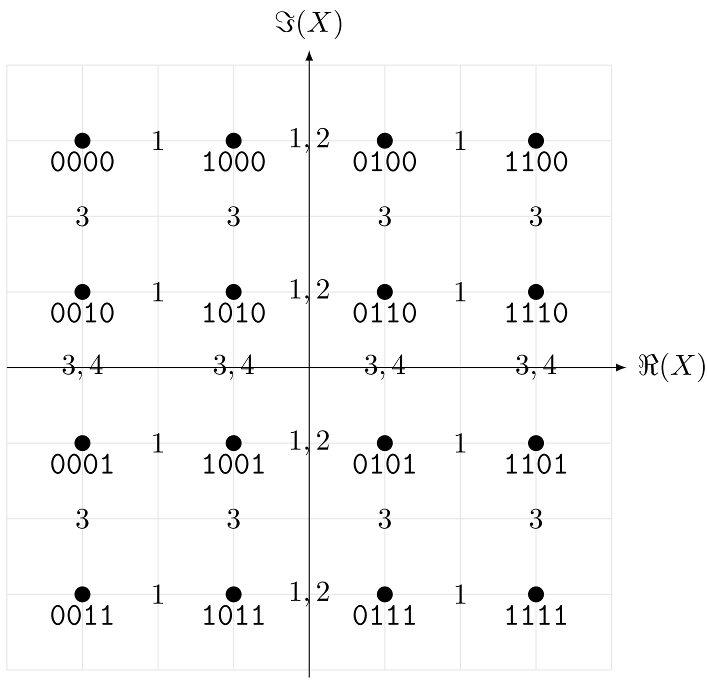
$$\bar{P}_e \leq \frac{3}{2\bar{x}} = \frac{15}{\bar{\gamma}}$$

by using the relationship $\bar{x} = \frac{1}{10}\bar{\gamma}$.

(b) From (a) it is clear that the diversity order is 1.

(c) We see that the $\bar{P}_\mathrm{e}$ for the 16-QAM constellation is much higher than the BPSK case. (Although we only have an upper bound on the error probability for the 16-QAM, we expect it is reasonably tight for high SNR since the squared Q-function term decays very fast.) The reason is that in the high SNR regime, the confusion with the nearest neighbors are the dominant error events, and obviously the minimum distance in 16-QAM ($2\sqrt{\mathcal{E}/10}$) is much smaller than that in BPSK ($2\sqrt{\mathcal{E}}$) with the same average power. However they are the same in terms of the diversity: in both cases the average error probability scales as $1/\bar{\gamma}$.

(d) Let us write out the *position(s)* of the different bit(s) between two adjacent symbols on the constellation. For example, between `0010` and `1010`, the first bit is different; and between `0010` and `0001`, the third and fourth bits are different. Hence it can happen that when the symbol `0010` is sent, the first bit is incorrectly estimated (when confused with the symbol `1010`), or the third and fourth bits are incorrectly estimated (when confused with the symbol `0001`.)



Now consider the bit error probability of the fourth bit. We see that the fourth bit may be incorrectly estimated if the symbols from the second and third rows are sent (i.e., from `0010` to `0001`). The error probability of the fourth bit is

$$\begin{aligned} P_{\mathrm{e},4} &= \Pr\{\text{symbol incorrectly estimated}|\text{symbol from } 2^{\text{nd}} \text{ or } 3^{\text{rd}} \text{ row}\} \\ &\quad \times \Pr\{\text{symbol from } 2^{\text{nd}} \text{ or } 3^{\text{rd}} \text{ row}\} \\ &= Q\left(\sqrt{\frac{|h|^2\mathcal{E}}{10N_0/2}}\right) \cdot \frac{1}{2} \end{aligned}$$

However notice that the error probability of the third bit is

$$
\begin{aligned}
P_{\mathrm{e},3} =& \Pr\{\text{symbol incorrectly estimated}|\text{symbol from 2}^{\text{nd}}\text{ or 3}^{\text{rd}}\text{ row}\} \\
&\qquad \times \Pr\{\text{symbol from 2}^{\text{nd}}\text{ or 3}^{\text{rd}}\text{ row}\} \\
&\quad + \Pr\{\text{symbol incorrectly estimated}|\text{symbol from 1}^{\text{st}}\text{ or 4}^{\text{th}}\text{ row}\} \\
&\qquad \times \Pr\{\text{symbol from 1}^{\text{st}}\text{ or 4}^{\text{th}}\text{ row}\} \\
=& 2Q\left(\sqrt{\frac{|h|^2\mathcal{E}}{10N_0/2}}\right)\cdot\frac{1}{2} + Q\left(\sqrt{\frac{|h|^2\mathcal{E}}{10N_0/2}}\right)\cdot\frac{1}{2}
\end{aligned}
$$

By the same token we have

$$
P_{\mathrm{e},2} = \frac{1}{2}Q\left(\sqrt{\frac{|h|^2\mathcal{E}}{10N_0/2}}\right)
$$

$$
P_{\mathrm{e},1} = \frac{3}{2}Q\left(\sqrt{\frac{|h|^2\mathcal{E}}{10N_0/2}}\right)
$$

The average bit error probability is

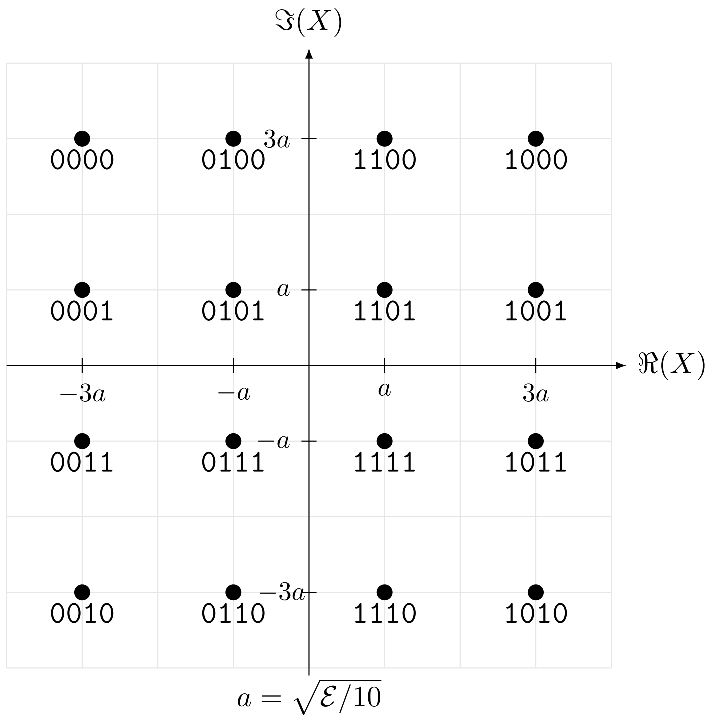
$$
P_{\mathrm{e,bit}} = Q\left(\sqrt{\frac{|h|^2\mathcal{E}}{10N_0/2}}\right)
$$

Similarly the average bit error probability average again over the fading channel gives

$$
\bar{P}_{\mathrm{e,bit}} \approx \frac{5}{\bar{\gamma}}
$$

which is one third of the symbol error probability obtained in (a). Obviously the diversity order is again 1.

(e) The following figure shows the so-called *Gray coded* 16-PAM. The feature of the Gray code is that each pair of adjacent symbols differ by only 1 bit.



The bit error probabilities are therefore

$$P_{\text{e},1} = P_{\text{e},3} = \frac{1}{2} Q\left(\sqrt{\frac{|h|^2 \mathcal{E}}{10 N_0/2}}\right)$$
$$P_{\text{e},2} = P_{\text{e},4} = Q\left(\sqrt{\frac{|h|^2 \mathcal{E}}{10 N_0/2}}\right)$$

and the average bit error probability is

$$P_{\text{e,bit}} = \frac{3}{4} Q\left(\sqrt{\frac{|h|^2 \mathcal{E}}{10 N_0/2}}\right)$$

which is lower than that of the first bit assignment. We can also show that for a fading channel, the average bit error probability is

$$\bar{P}_{\text{e,bit}} \approx \frac{15}{4\bar{\gamma}}$$

which is a quarter of the symbol error probability we have seen in part (a). This is logical, since

$$\begin{aligned}\bar{P}_{\text{e,bit}} &= \frac{1}{\text{number of total bits}} \mathbb{E}[\text{erroneous bits}] \\ &= \frac{1}{4}(1 \cdot \Pr\{\text{1 bit wrong}\} + 2 \cdot \Pr\{\text{2 bits wrong}\} + \ldots)\end{aligned}$$

In the Gray constellation, the probability of getting more than one bits wrong is very small, since any adjacent symbols differ by only one bit. Hence the error probability $\Pr\{\text{1 bit wrong}\}$ is essentially the symbol error probability, which gives

$$\bar{P}_{\text{e,bit}} = \frac{1}{4} P_{\text{e,symbol}}$$

Solution 3. Let D denote the sum of these hard decisions. Without loss of generality, we assume 0 is transmitted. Then, the error probability is simply $\overline{P_{\text{e}}} = \Pr\{D \geq L/2\}$. Since each channel is independent, D satisfies the well-known Binomial distribution:

$$\overline{P_{\text{e}}} = \Pr\{D \geq L/2\} = \sum_{d=\lceil L/2 \rceil}^{L} \binom{L}{d} p^d (1-p)^{L-d}$$

where p is defined as $p = \Pr\{Y[n] \geq 0\}$ for any n.

If you are not sure about this, the following is a formal derivation.

$$
\begin{aligned}
\overline{P_e} &= \Pr\{D \geq L/2\} \\
&= \sum_{d=\lceil L/2 \rceil}^{L} \Pr\{D = d\} \\
&= \sum_{d=\lceil L/2 \rceil}^{L} \Pr\left\{\sum_{n=1}^{L} \mathbf{1}_{\{Y[n]\geq 0\}} = d\right\} \\
&= \sum_{d=\lceil L/2 \rceil}^{L} \sum_{(b_1,\cdots,b_L)\in\{0,1\}^L} \Pr\left\{\sum_{n=1}^{L} \mathbf{1}_{\{Y[n]\geq 0\}} = d \middle| \mathbf{1}_{\{Y[1]\geq 0\}} = b_1, \cdots, \mathbf{1}_{\{Y[L]\geq 0\}} = b_L\right\} \\
&\quad \cdot \Pr\left\{\mathbf{1}_{\{Y[1]\geq 0\}} = b_1, \cdots, \mathbf{1}_{\{Y[L]\geq 0\}} = b_L\right\} \\
&\overset{(1)}{=} \sum_{d=\lceil L/2 \rceil}^{L} \sum_{\substack{(b_1,\cdots,b_L)\in\{0,1\}^L \\ \text{s.t. } \sum_{n=1}^{L} b_n = d}} \prod_{n=1}^{L} \Pr\left\{\mathbf{1}_{\{Y[n]\geq 0\}} = b_n\right\} \\
&\overset{(2)}{=} \sum_{d=\lceil L/2 \rceil}^{L} \sum_{\substack{(b_1,\cdots,b_L)\in\{0,1\}^L \\ \text{s.t. } \sum_{n=1}^{L} b_n = d}} p^d (1-p)^{L-d} \\
&= \sum_{d=\lceil L/2 \rceil}^{L} \binom{L}{d} p^d (1-p)^{L-d},
\end{aligned}
$$

where (1) follows since $\{H[n], Z[n]\}$ are independent, (2) follows since $\{Y[n]\}$ are identically distributed, and

$$
\begin{aligned}
p &= \Pr\{Y[1] \geq 0\} \\
&= \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right),
\end{aligned}
$$

where $\overline{\gamma} = 8\mathcal{E}/N_0$.

To show the diversity of this scheme, we want lower and upper bound the average error probability $\overline{P_e}$. The error probability can be lower bounded as (note that $p \leq 1/2$)

$$
\begin{aligned}
\overline{P_e} &\geq \binom{L}{\lceil L/2 \rceil} p^{\lceil L/2 \rceil} (1-p)^{L-\lceil L/2 \rceil} \\
&> p^{\lceil L/2 \rceil} \left(\frac{1}{2}\right)^{L-\lceil L/2 \rceil},
\end{aligned}
$$

To upper bound the error probability, we will use the following simple observation:

$$
\max\{a_i x_i\} \leq \sum a_i x_i \leq \max\{x_i\} \times \sum a_i,
$$

for all $a_i \geq 0$, $x_i \geq 0$. Hence $\overline{P_e}$ is upper bounded as

$$
\begin{aligned}
\overline{P_e} &\leq \left(\max_{d\geq\lceil L/2 \rceil} p^d (1-p)^{L-d}\right) \sum_{d=\lceil L/2 \rceil}^{L} \binom{L}{d} \\
&< p^{\lceil L/2 \rceil} (1-p)^{L-\lceil L/2 \rceil} 2^L \\
&\leq p^{\lceil L/2 \rceil} 2^L
\end{aligned}
$$

For large $\overline{\gamma}$, we can approximate the error probability as $p \approx \frac{1}{2\overline{\gamma}}$. Then, we have

$$\left(\frac{1}{2}\right)^{L}\left(\frac{1}{\overline{\gamma}}\right)^{\lceil L/2 \rceil} \leq \overline{P_e} \leq 2^{L-\lceil L/2 \rceil}\left(\frac{1}{\overline{\gamma}}\right)^{\lceil L/2 \rceil},$$

and thus

$$\lceil L/2 \rceil - \frac{(L - \lceil L/2 \rceil)\log 2}{\log \overline{\gamma}} \leq -\frac{\log \overline{P_e}}{\log \overline{\gamma}} \leq \lceil L/2 \rceil + \frac{L \log 2}{\log \overline{\gamma}}.$$

Finally, letting $\overline{\gamma} \to \infty$, we see that the diversity order of the hard-decision decoding is $\lceil L/2 \rceil$.