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Handout 13 Advanced Digital Communications
Solutions to Homework 5 Oct. 31, 2016

SOLUTION 1. We first compute $D(z)$ as follows

$$\begin{aligned}
\sum_{k:\text{odd}} 2^{-\frac{|k|-1}{2}} z^{-k} &= \sum_{k:\text{odd},k>0} 2^{-\frac{|k|-1}{2}} z^{-k} + \sum_{k:\text{odd},k<0} 2^{-\frac{|k|-1}{2}} z^{-k} \\
&\overset{(*)}{=} \sum_{m=0}^{\infty} 2^{-m} z^{-(2m+1)} + \sum_{m=0}^{\infty} 2^{-m} z^{2m+1} \\
&= z^{-1} \sum_{m=0}^{\infty} (\frac{z^{-2}}{2})^m + z \sum_{m=0}^{\infty} (\frac{1}{2z^{-2}})^m \\
&= \frac{z^{-1}}{1-\frac{z^{-2}}{2}} + \frac{1}{z^{-1}(1-\frac{1}{2z^{-2}})} \\
&= \frac{z^{-1}}{1-\frac{z^{-2}}{2}} + \frac{z}{1-\frac{z^2}{2}}
\end{aligned}$$

where in $(*)$ we have taken $m = \frac{|k|-1}{2}$. Similarly

$$\begin{aligned}
\sum_{k:\text{even}} 2^{-\frac{|k|}{2}} z^{-k} &= \sum_{k:\text{even},k\geq 0} 2^{-\frac{|k|}{2}} z^{-k} + \sum_{k:\text{even},k<0} 2^{-\frac{|k|}{2}} z^{-k} \\
&\overset{(*)}{=} \sum_{m=0}^{\infty} 2^{-m} z^{-2m} + \sum_{m=1}^{\infty} 2^{-m} z^{2m} \\
&= \sum_{m=0}^{\infty} (\frac{z^{-2}}{2})^m + \sum_{m=1}^{\infty} (\frac{1}{2z^{-2}})^m \\
&= \frac{1}{1-\frac{z^{-2}}{2}} + \frac{\frac{1}{2z^{-2}}}{1-\frac{1}{2z^{-2}}} \\
&= \frac{1}{1-\frac{z^{-2}}{2}} + \frac{\frac{z^2}{2}}{1-\frac{z^2}{2}}
\end{aligned}$$

where $(*)$ follows again by taking $m = \frac{|k|}{2}$. Consequently,

$$\begin{aligned}
D(z) &= \sum_{k=-\infty}^{\infty} 2^{-\frac{|k|-1}{2}} z^{-k} = \left[\frac{z^{-1}}{1-\frac{z^{-2}}{2}} + \frac{z}{1-\frac{z^2}{2}}\right] + \frac{5}{3}\left[\frac{1}{1-\frac{z^{-2}}{2}} + \frac{\frac{z^2}{2}}{1-\frac{z^2}{2}}\right] \\
&= \frac{\frac{5}{3}+z^{-1}}{1-\frac{z^{-2}}{2}} + \frac{\frac{5}{6z^{-2}}+z}{1-\frac{1}{2z^{-2}}} \\
&= \frac{\frac{5}{3}-\frac{5}{6z^{-2}}+z^{-1}-\frac{1}{2z^{-1}}+\frac{5}{6z^{-2}}-\frac{5}{12}+z-\frac{z^{-1}}{2}}{(1-\frac{1}{2}z^{-2})(1-\frac{1}{2}z^2)} \\
&= \frac{\frac{5}{4}+\frac{1}{2}z^{-1}+\frac{1}{2}z}{(1-\frac{1}{2}z^{-2})(1-\frac{1}{2}z^2)} \\
&= \frac{(1+\frac{1}{2}z^{-1})(1+\frac{1}{2}z)}{(1-\frac{1}{2}z^{-2})(1-\frac{1}{2}z^2)} \\
&= F(z)F^*(1/z^*).
\end{aligned}$$

Hence, the whitening filter will be

$$D_W(z) = \frac{1}{F^*(1/z^*)} = \frac{1 - \frac{1}{2}z^2}{1 + \frac{1}{2}z}$$

Solution 2.

(a) For the given example, we have that

$$D(z) \;=\; \alpha + \beta(z + z^{-1})$$

or, in the frequency domain,

$$D(f) \;=\; \alpha + \beta(e^{j2\pi f} + e^{-j2\pi f}) = \alpha + 2\beta \cos 2\pi f.$$

The zero-forcing filter is simply $\frac{1}{D(f)}$. Hence, the power spectral density of the equivalent noise $\tilde{V}[n]$ (see Equation (4.13) of the lecture notes) is

$$S_{\tilde{V}}(f) \;=\; \frac{1}{|D(f)|^2}\frac{N_0}{2}D(f) = \frac{N_0}{2}\frac{1}{D(f)}.$$

As we have done several times in class, we know that the variance of this noise can be calculated by observing

$$\mathbb{E}[\tilde{V}^2[n]] \;=\; R_{\tilde{V}}[0] = \int_{-1/2}^{1/2} S_{\tilde{V}}(f)df.$$

Using the hints in the problem statement, we can actually evaluate this integral:

$$\begin{aligned}
\mathbb{E}[\tilde{V}^2[n]] &= \frac{N_0}{2}\int_{-1/2}^{1/2} \frac{1}{\alpha + 2\beta\cos 2\pi f}df \\
&= \frac{N_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\alpha + 2\beta\cos w}dw \\
&= \frac{N_0}{2\sqrt{\alpha^2 - 4\beta^2}}
\end{aligned}$$

(b) We first prove the following:

$$\begin{aligned}
\sigma^2_{\text{LMMSE}} &= \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])^2\right] \\
&= \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])(I[n] - \hat{I}_{\text{LMMSE}}[n])\right] \\
&= \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])I[n]\right] - \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])\hat{I}_{\text{LMMSE}}[n])\right] \\
&= \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])I[n]\right] - \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])\sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]U[n-k]\right] \\
&= \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])I[n]\right] - \sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]\mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])U[n-k]\right] \qquad (1) \\
&= \mathbb{E}\left[(I[n] - \hat{I}_{\text{LMMSE}}[n])I[n]\right] \\
&= \mathbb{E}\left[(I[n])^2\right] - \mathbb{E}\left[\hat{I}_{\text{LMMSE}}[n]I[n]\right],
\end{aligned}$$

where, in (1), we can apply the orthogonality principle to conclude that the second part of the expression is zero. But then, we also trivially observe that

$$\mathbb{E}\left[(I[n])^2\right] \;=\; \mathcal{E}.$$

So, the only difficulty is the term $\mathbb{E}\left[\hat{I}_{\text{LMMSE}}[n]I[n]\right]$. To tackle this term, let us first write out:

$$\begin{aligned}\hat{I}_{\text{LMMSE}}[n] &= \sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]U[n-k] \\ &= \sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]\left(\sum_{\ell=-\infty}^{\infty} I[n-k-\ell]d[\ell]+V[n-k]\right).\end{aligned}$$

But then,

$$\begin{aligned}\mathbb{E}\left[\hat{I}_{\text{LMMSE}}[n]I[n]\right] &= \mathbb{E}\left[\left(\sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]\sum_{\ell=-\infty}^{\infty} I[n-k-\ell]d[\ell]+V[n-k]\right)I[n]\right] \\ &= \sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]\sum_{\ell=-\infty}^{\infty} d[\ell]\mathbb{E}\left[I[n-k-\ell]I[n]\right] \\ &\quad + \sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]\,\mathbb{E}\left[V[n-k]I[n]\right]\end{aligned}$$

Now, clearly, the last term is zero since the signal and the noise are uncorrelated (in fact, independent) and of mean zero. Moreover, because we assume that $I[n]$ is i.i.d. we have that $\mathbb{E}\left[I[n-k-\ell]I[n]\right]=\mathcal{E}\delta[k+\ell]$. Hence,

$$\begin{aligned}\mathbb{E}\left[\hat{I}_{\text{LMMSE}}[n]I[n]\right] &= \mathcal{E}\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_{\text{LMMSE}}[k]d[\ell]\delta[k+\ell] \\ &= \mathcal{E}\sum_{k=-\infty}^{\infty} a_{\text{LMMSE}}[k]d[-k].\end{aligned}$$

Now, how do we calculate this sum? One way is to observe that it is simply the convolution of $a_{\text{LMMSE}}[k]$ with $d[k]$, evaluated at zero. In general, let us define the new signal $b[n]=(a_{\text{LMMSE}}*d)[k]$, and hence, in terms of the signal $b[n]$, we have that

$$\mathbb{E}\left[\hat{I}_{\text{LMMSE}}[n]I[n]\right] \;=\; \mathcal{E}b[0].$$

All that's left to do is to find $b[0]$. Let us study the Fourier transform of the signal $b[k]$. From the convolution property, we have

$$B(f) \;=\; A_{\text{LMMSE}}(f)D(f)=\frac{\mathcal{E}D(f)}{\mathcal{E}D(f)+\frac{N_0}{2}},$$

where we have used the formula for $A_{\text{LMMSE}}(f)$ as derived in class for the special case when the signal $I[n]$ is i.i.d. Now, trivially from the inverse Fourier transform formula, we know that

$$\begin{aligned}
b[0] &= \int_{-1/2}^{1/2} B(f) df \\
&= \int_{-1/2}^{1/2} \frac{\mathcal{E}D(f)}{\mathcal{E}D(f) + \frac{N_0}{2}} df \\
&= \int_{-1/2}^{1/2} \left(1 - \frac{\frac{N_0}{2}}{\mathcal{E}D(f) + \frac{N_0}{2}}\right) df \\
&= 1 - \int_{-1/2}^{1/2} \frac{\frac{N_0}{2}}{\mathcal{E}D(f) + \frac{N_0}{2}} df.
\end{aligned}$$

We have already found that $D(f) = \alpha + 2\beta \cos 2\pi f$. Hence,

$$b[0] = 1 - \int_{-1/2}^{1/2} \frac{\frac{N_0}{2}}{\mathcal{E}(\alpha + 2\beta \cos 2\pi f) + \frac{N_0}{2}} df.$$

Finally, using the hint given in the problem statement, we find

$$b[0] = 1 - \frac{\frac{N_0}{2\mathcal{E}}}{\sqrt{(\frac{N_0}{2\mathcal{E}} + \alpha)^2 - 4\beta^2}}$$

Combining our expressions, we conclude that

$$\mathbb{E}[(\hat{I}_{\text{LMMSE}}[n] - I[n])^2] = \mathcal{E}(1 - b[0]) = \frac{N_0}{2} \frac{1}{\sqrt{(\frac{N_0}{2\mathcal{E}} + \alpha)^2 - 4\beta^2}}. \tag{2}$$

(c) The Z-transform of the matched filter is $D(1/z) = \alpha + \beta(z^{-1} + z) = D(Z)$. Thus, in the time domain we have that:

$$\begin{aligned}
I_{\text{MF}}[n] &= (d * d * I)[n] + (d * V)[n] \\
&= I[n](\alpha^2 + 2\beta^2) + 2\alpha\beta(I[n-1] + I[n+1]) + \beta^2(I[n-2] + I[n+2]) \\
&\quad + \alpha V[n] + \beta(V[n-1] + V[n+1])
\end{aligned}$$

The matched filter output is thus indeed nothing more than a linear sum of the current symbol and some interference and noise components. Then, we find the effective noise variance by normalizing the desired symbol $I[n]$ and finding the variance of the undesired components:

$$\begin{aligned}
&\frac{1}{g_0[n]} I_{\text{MF}}[n] = I[n] \\
&+ \underbrace{\frac{2\alpha\beta(I[n-1] + I[n+1]) + \beta^2(I[n-2] + I[n+2]) + \alpha V[n] + \beta(V[n-1] + V[n+1])}{\alpha^2 + 2\beta^2}}_{\text{undesired}}
\end{aligned}$$

Because all the signal and noise terms are independent, we can calculate their power separately and sum them up:

$$E[((d * V)[n])^2] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\alpha + 2\beta \cos w)^2 \frac{N_0}{2} (\alpha + 2\beta \cos w) dw = (\alpha^3 + 6\alpha\beta^2) \frac{N_0}{2}$$

and

$$P_{\text{Interference}} = 2 \times 4\alpha^2\beta^2\mathcal{E} + 2 \times \beta^4\mathcal{E} = \mathcal{E}(8\alpha^2\beta^2 + 2\beta^4)$$

The equivalent noise variance is thus

$$\sigma^2_{\text{MF}} = \frac{\mathcal{E}(8\alpha^2\beta^2 + 2\beta^4) + \frac{N_0}{2}(\alpha^3 + 6\alpha\beta^2)}{(\alpha^2 + 2\beta^2)^2}$$

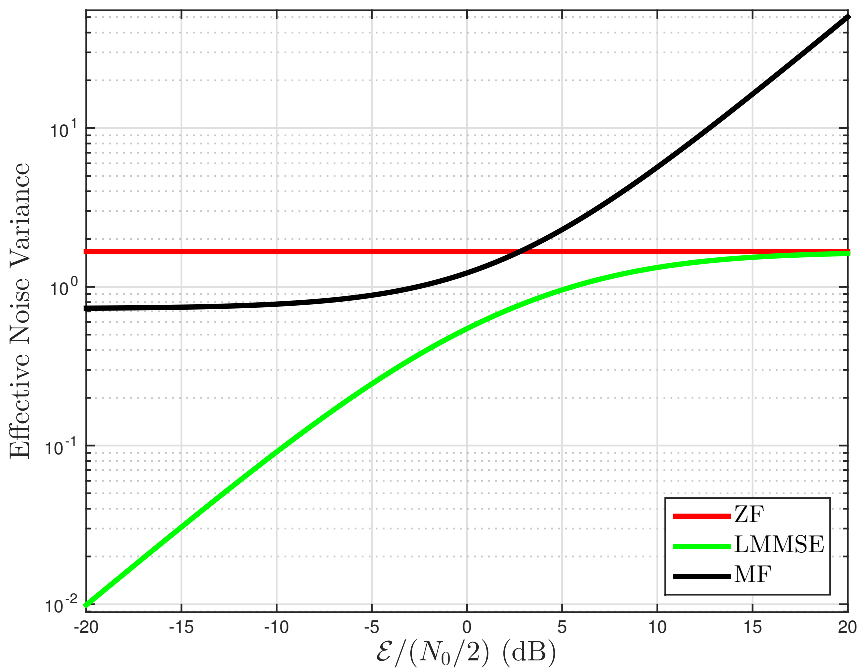
(d) For this choice of parameters we have:

$$\sigma^2_{\text{ZF}} = \frac{N_0}{2\sqrt{\alpha^2 - 4\beta^2}} = \frac{1}{\sqrt{0.36}}$$

$$\sigma^2_{\text{LMMSE}} = \frac{N_0}{2}\frac{1}{\sqrt{(\frac{N_0}{2\mathcal{E}} + \alpha)^2 - 4\beta^2}} = \frac{1}{\sqrt{(\frac{1}{\mathcal{E}} + 1)^2 - 0.64}}$$

$$\sigma^2_{\text{MF}} = \frac{\mathcal{E}(8\alpha^2\beta^2 + 2\beta^4) + \frac{N_0}{2}(\alpha^3 + 6\alpha\beta^2)}{(\alpha^2 + 2\beta^2)^2} = \frac{1.3312\mathcal{E} + 1.96}{1.7424}$$

The plot is given below:



Solution 3. First consider estimating the signal $I[n]$ from user 1. The orthogonality principle implies the optimal estimator must satisfy

$$\mathbb{E}\left[(\hat{I}[n] - I[n])Y^*[n-k]\right] = 0, \qquad \forall k = -1, 0, 1.$$

(Note that we would like to estimate $I[n]$ given $Y[n+1]$, $Y[n]$, and $Y[n-1]$.) Since $\hat{I}[n] = \sum_{\ell=-1}^{1} a_\ell Y[n-\ell]$, the above simplifies to

$$\mathbb{E}\left[\sum_{\ell=-1}^{1} a_\ell Y[n-\ell]Y^*[n-k]\right] = \mathbb{E}[I[n]Y^*[n-k]]$$

or equivalently

$$\sum_{\ell=-1}^{1} a_\ell R_Y[k-\ell] = \mathbb{E}[I[n]Y^*[n-k]] := R_{IY}[k], \tag{3}$$

for all $k = -1, 0, 1$. It is easy to verify that

$$R_Y[m] = \mathbb{E}\big[Y[n+m]Y^*[n]\big] = 8\delta[m] - \delta[m+1] - \delta[m-1],$$

and (even more easily)

$$R_{IY}[m] = \mathbb{E}\big[I[n+m]Y^*[n]\big] = \delta[m] - 2\delta[m+1].$$

Writing out (3) explicitly for $k = -1, 0, 1$ in the matrix form gives us

$$\begin{pmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 8 \end{pmatrix} \begin{pmatrix} a_{-1} \\ a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Solving the above equations we get the optimal coefficients

$$a_{-1} = -0.2379, a_0 = 0.0968, a_1 = 0.0121$$

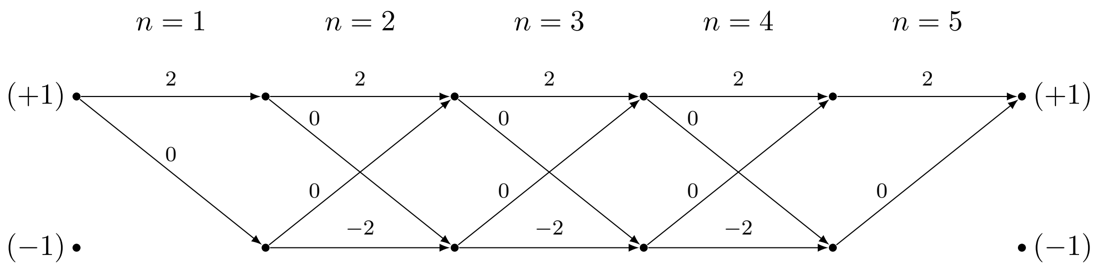
Using the same argument for user, we get the linear system

$$\begin{pmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 81 \end{pmatrix} \begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

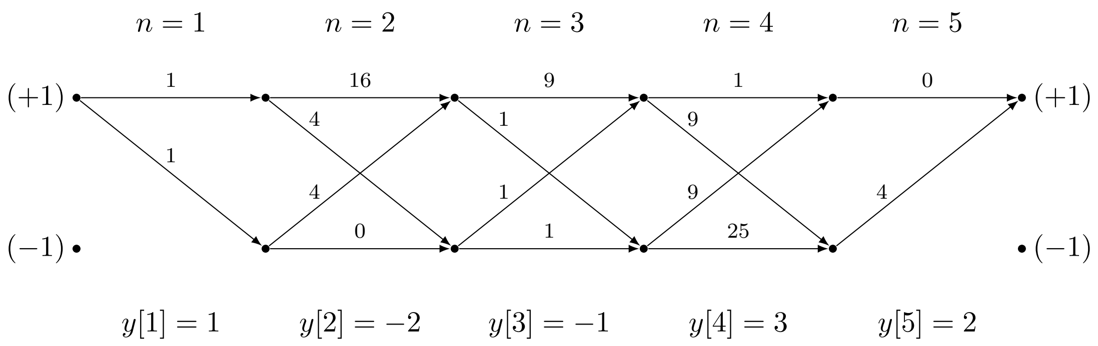
which gives the optimal coefficients

$$b_{-1} = 0.1431, b_0 = 0.1452, b_1 = 0.0181$$

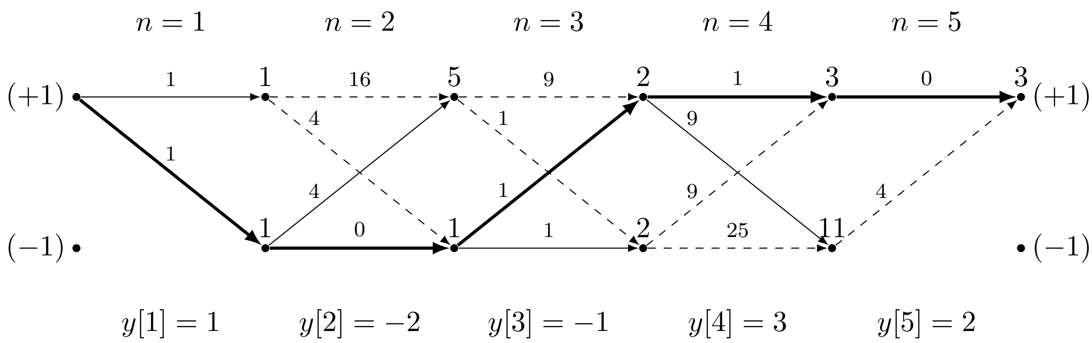
Solution 4. Let us first draw the trellis (here the edges are labeled with the value of noiseless channel output):



Next, we can label the edges with the metric $|y[n] - (x[n] + x[n-1])|^2$:

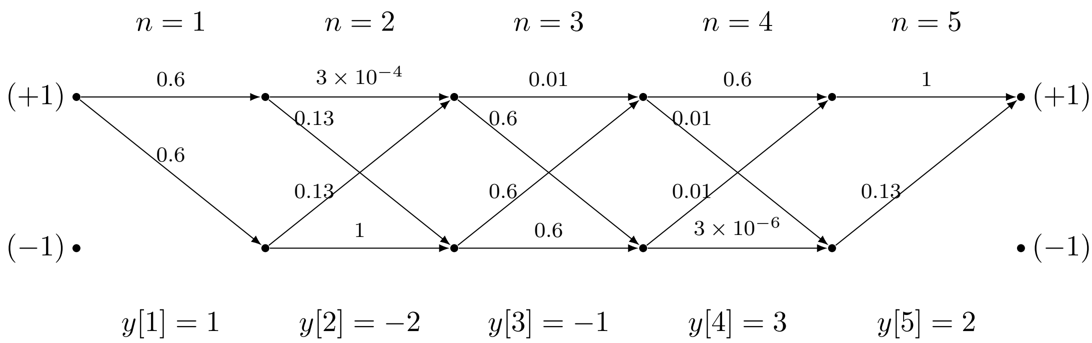


The MLSE (Viterbi) decoder will choose the following path on the trellis:



and estimates the input sequence to be $(-1, -1, +1, +1, +1)$.

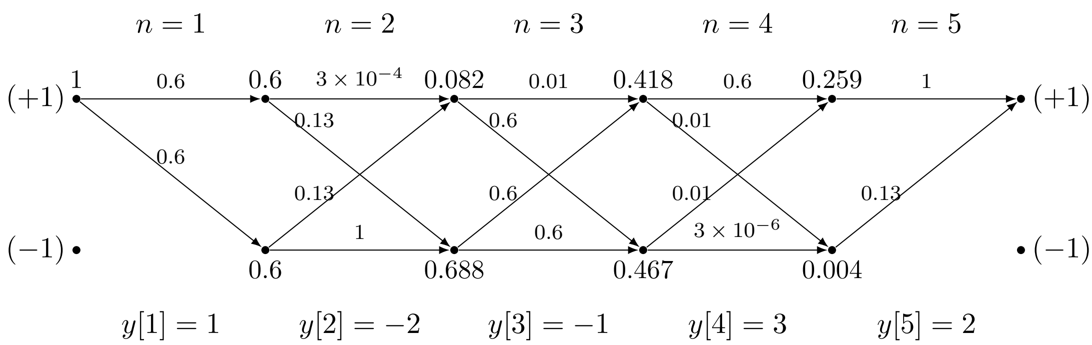
For the BCJR algorithm we first relabel the edges with $\gamma_n(x[n-1], x[n]) = \exp\{-\frac{1}{2}|y[n]-(x[n]+x[n-1])|^2\}$ (the numbers you see on the trellis are rounded off):



The *forward recursion* phase of the algorithm assigns to each vertex the value of

$$\alpha_n(x) = \sum_{x' \in \{\pm 1\}} \alpha_{n-1}(x')\gamma_n(x, x')$$

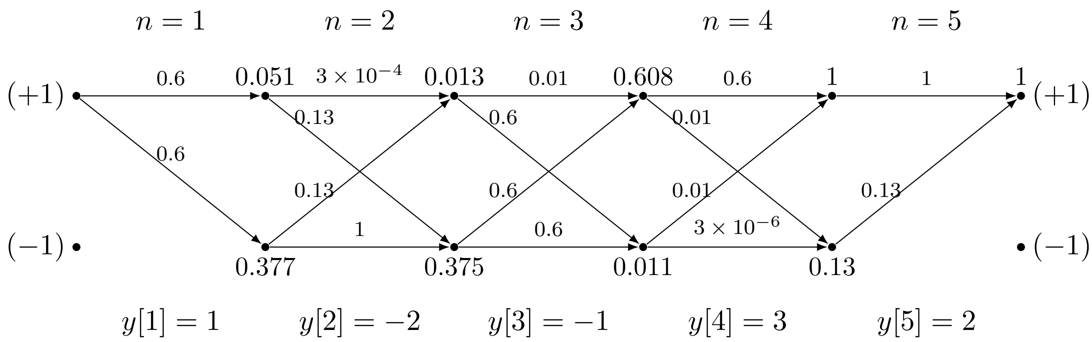
as follows,



while the *backward recursion* phase assigns to each vertex the value of

$$\beta_n(x) = \sum_{x' \in \{\pm 1\}} \gamma_n(x, x')\beta_{n+1}(x)$$

as follows



By computing $\text{score}_n(x) = \alpha_n(-x)\gamma_n(-x,x)\beta_{n+1}(x) + \alpha_n(x)\gamma_n(x,x)\beta_{n+1}(x)$ we obtain the following result:

n	1	2	3	4	5
$+1$	0.0306	0.00101	0.2515	0.2555	
-1	0.2262	0.25425	0.0051	0.0005	
$\hat{x}[n]$	-1	-1	$+1$	$+1$	$+1$

SOLUTION 5.

(a) (i)

$$\mathbb{E}[\hat{X}|X=x] = \mathbb{E}[\boldsymbol{a}^T\boldsymbol{h}X + \boldsymbol{a}^T\boldsymbol{Z}|X=x] = \mathbb{E}[\boldsymbol{a}^T\boldsymbol{h}X|X=x] \Leftrightarrow \boldsymbol{a}^T\boldsymbol{h}x = x \Leftrightarrow \boldsymbol{a}^T\boldsymbol{h} = 1$$

(ii) First observe that:

$$\hat{X} = \boldsymbol{a}^T\boldsymbol{Y} = x - \boldsymbol{a}^T\boldsymbol{Z}$$

then:

$$x - \hat{x} = -\boldsymbol{a}^T\boldsymbol{Z}$$

Therefore:

$$\mathbb{E}[|X-\hat{X}|^2] = \mathbb{E}[(\boldsymbol{a}^T\boldsymbol{Z})^2] = \boldsymbol{a}^T\,\mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}^T]\boldsymbol{a} = \boldsymbol{a}^T I\boldsymbol{a} = \boldsymbol{a}^T\boldsymbol{a} = \sigma^2_{\text{unbiased}}$$

So we need to minimize $\boldsymbol{a}^T\boldsymbol{a}$ such that $\boldsymbol{a}^T\boldsymbol{h} = 1$. The solution to the minimization problem is $\boldsymbol{a} = (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T$. In this case $\sigma^2_{\text{unbiased}} = \boldsymbol{a}^T\boldsymbol{a} = (\boldsymbol{h}^T\boldsymbol{h})^{-1}$.
We can indeed verify the solution by observing that:

$$(\boldsymbol{a}^T\boldsymbol{a})(\boldsymbol{h}^T\boldsymbol{h}) \geq |\boldsymbol{a}^T\boldsymbol{h}|^2 = 1 \Rightarrow \boldsymbol{a}^T\boldsymbol{a} \geq (\boldsymbol{h}^T\boldsymbol{h})^{-1}$$

with equality when $\boldsymbol{a}$ is chosen as proposed.

(b) Let $c := \boldsymbol{a}^T\boldsymbol{h}$, thus,

$$\hat{X} = \boldsymbol{a}^T\boldsymbol{Y} = cX - \boldsymbol{a}^T\boldsymbol{Z}$$

then:

$$X - \hat{X} = (1-c)X - \boldsymbol{a}^T\boldsymbol{Z}$$

Therefore,

$$\mathbb{E}[|X-\hat{X}|^2] = \mathbb{E}\left[\left[(1-c)X - (\boldsymbol{a}^T\boldsymbol{Z})\right]^2\right] = (1-c)^2\mathcal{E} + \boldsymbol{a}^T\boldsymbol{a} = \sigma^2_{\min}(c)$$

Since $(1-c)^2\mathcal{E}$ is fixed we just need to minimize $\boldsymbol{a}^T\boldsymbol{a}$ such that $\boldsymbol{a}^T\boldsymbol{h} = c$. The solution to the minimization problem is $\boldsymbol{a} = c(\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T$. In this case:

$$\sigma^2_{\min}(c) = (\boldsymbol{h}^T\boldsymbol{h})^{-1}c^2 + (c-1)^2\mathcal{E}$$

We can find the c that minimizes $\sigma^2_{\text{min}}(c)$ by finding the zero of the derivative. We obtain:

$$c = \frac{\mathcal{E}}{(\boldsymbol{h}^T\boldsymbol{h})^{-1} + \mathcal{E}}$$

The minimal $\sigma^2_{\text{min}}(c)$ is therefore:

$$\sigma^2_{\text{min}} = \frac{\mathcal{E}}{\boldsymbol{h}^T\boldsymbol{h}\mathcal{E} + 1}$$

(c) In the first case we have:

$$\frac{\mathcal{E}}{\sigma^2_{\text{unbiased}}} = \mathcal{E}\boldsymbol{h}^T\boldsymbol{h}$$

In the second case:

$$\frac{\mathcal{E}}{\sigma^2_{\text{min}}} = \mathcal{E}\boldsymbol{h}^T\boldsymbol{h} + 1$$

(d) We notice that in both cases $\boldsymbol{a}^T = c \cdot (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T$ with $c = 1$ for the first part. The probability of making an error is:

$$\begin{aligned}
\Pr\{\hat{X} = k | X = -k\} &= \Pr\{\hat{X} = 1 | X = -1\} = \Pr\{-\boldsymbol{a}^T\boldsymbol{h} + \boldsymbol{a}^T\boldsymbol{Z} > 0\} \\
&= \Pr\{\boldsymbol{a}^T\boldsymbol{Z} > \boldsymbol{a}^T\boldsymbol{h}\} = \Pr\{c \cdot (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T\boldsymbol{Z} > c \cdot (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T\boldsymbol{h}\} = \\
&= \Pr\{(\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T\boldsymbol{Z} > 1\}
\end{aligned}$$

which is independent of c. This means that both estimators have the same error probability.