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School of Computer and Communication Sciences

Handout 7 Advanced Digital Communications
Solutions to Homework 2 Oct. 10, 2016

SOLUTION 1.

(a) Let us first derive the mean:

$$\boldsymbol{m_Z} = \mathbb{E}[A\boldsymbol{Y}] = A\,\mathbb{E}[\boldsymbol{Y}] = A\boldsymbol{m}.$$

Then the transformed covariance follows as

$$\begin{aligned} C_{\boldsymbol{Z}} &= \mathbb{E}[(\boldsymbol{Z}-\boldsymbol{m_Z})(\boldsymbol{Z}-\boldsymbol{m_Z})^H] \\ &= \mathbb{E}[A(\boldsymbol{Y}-\boldsymbol{m})\,(A(\boldsymbol{Y}-\boldsymbol{m}))^H] \\ &= A\,\mathbb{E}[(\boldsymbol{Y}-\boldsymbol{m})(\boldsymbol{Y}-\boldsymbol{m})^H]A^H \\ &= ACA^H. \end{aligned}$$

(b) First, if α or β is zero, then Z_1 is clearly Gaussian. In the following, we assume $\alpha \neq 0 \neq \beta$. We show that $Z_1 = \alpha Y_1 + \beta Y_2$ is a Gaussian random variable by demonstrating that the pdf of Z_1 is of Gaussian form as well. We change variables from (Y_1, Y_2) to Z_1 by integrating the joint-pdf (Y_1, Y_2) over $\mathbb{R}^2$ with an indicator-function that yields 1 if $\alpha y_1 + \beta y_2 = z_1$ as follows

$$\begin{aligned} f_{Z_1}(z_1) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_{Y_1,Y_2}(y_1,y_2)\mathbb{1}\{\alpha y_1 + \beta y_2 = z_1\}\,dxdy \\ &\overset{(a)}{=} \frac{1}{\alpha\beta}\int_{\mathbb{R}}\int_{\mathbb{R}} f_{Y_1,Y_2}\left(\frac{u}{\alpha},\frac{w}{\beta}\right)\mathbb{1}\{u+w=z_1\}\,dudw \\ &= \frac{1}{\alpha\beta}\int_{\mathbb{R}} f_{Y_1,Y_2}\left(\frac{u}{\alpha},\frac{z_1-u}{\beta}\right)du \\ &\overset{(b)}{=} \frac{1}{\alpha\beta}\int_{\mathbb{R}} f_{Y_1}\left(\frac{u}{\alpha}\right) f_{Y_2}\left(\frac{z_1-u}{\beta}\right)du \\ &= \frac{1}{\sqrt{(2\pi)^2\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}}\int \exp\left(-\frac{\frac{u^2}{\alpha^2}}{2\sigma_{Y_1}^2}\right)\exp\left(-\frac{\frac{(z_1-u)^2}{\beta^2}}{2\sigma_{Y_2}^2}\right)du, \end{aligned}$$

where (a) follows by a change of variable $u = \alpha y_1, w = \beta y_2$ and (b) follows from the fact that Y_1 and Y_2 are independent.

Although we see the variables u and z_1, this expression is a *function* only over z_1. if one would work this out, u would ultimately be removed completely by the integration. At this point we cannot solve this integral, though. One trick to finding a closed form solution is to rewrite difficult integrals to other integrals of whom we do happen to know the closed form solution. To that end, let us group the two exponents and do some rewriting. To be precise, we are going to extract some expression that only contains z_1 and a residual over u that we do know how to solve.

$$
\begin{aligned}
&-\frac{u^2}{2\alpha^2\sigma_{Y_1}^2}-\frac{(z_1-u)^2}{2\beta^2\sigma_{Y_2}^2}=-\frac{u^2}{2\alpha^2\sigma_{Y_1}^2}-\frac{1}{2\beta^2\sigma_{Y_2}^2}\left(z_1^2-2uv+u^2\right)\\
&=-\frac{1}{2\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}\left(\alpha^2\sigma_{Y_1}^2z_1^2-2\alpha^2\sigma_{Y_1}^2uv+u^2\left(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2\right)\right)\\
&\overset{(a)}{=}-\frac{1}{2\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}\left(\frac{\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}z_1^2+\left(\frac{\alpha^2\sigma_{Y_1}^2}{\sqrt{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}}z_1-\sqrt{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}u\right)^2\right)\\
&=-\frac{1}{2(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2)}z_1^2-\frac{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}{2\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}\left(\frac{\alpha^2\sigma_{Y_1}^2}{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}z_1-u\right)^2.
\end{aligned}
$$

Now step (a) may be a bit hard to see. If you wish to see why this is true, try to write out the expressions and work your way back. The trick is that we *force* the quadratic expression on the right side and we alter the coefficient of z_1^2 accordingly to make sure the expression is still the same.

If we fill this back into our integral, we find

$$
\begin{aligned}
f_{Z_1}(z_1)&=\frac{1}{2\pi\sqrt{\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}}\int\exp\left(-\frac{\frac{u^2}{\alpha^2}}{2\sigma_{Y_1}^2}\right)\exp\left(-\frac{\frac{(z_1-u)^2}{\beta^2}}{2\sigma_{Y_2}^2}\right)du\\
&=\frac{1}{2\pi\sqrt{\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}}\exp\left(-\frac{z_1^2}{2(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2)}\right)\\
&\int\exp\left(-\frac{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}{2\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}\left(\frac{\alpha^2\sigma_{Y_1}^2}{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}z_1-u\right)^2\right)du.
\end{aligned}
$$

If we play around with the constant at the front, we can find a -tedious, but- standard Gaussian integral over the entire real line, which is known to integrate to 1, despite its ugly offset mean!

$$
\begin{aligned}
f_{Z_1}(z_1)&=\frac{1}{2\pi}\sqrt{\frac{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}{(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2)\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}}\exp\left(-\frac{z_1^2}{2(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2)}\right)\\
&\int\exp\left(-\frac{\left(\frac{\alpha^2\sigma_{Y_1}^2}{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}z_1-u\right)^2}{\frac{2\alpha^2\sigma_{Y_1}^2\beta^2\sigma_{Y_2}^2}{\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2}}\right)du\\
&=\frac{1}{\sqrt{2\pi\left(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2\right)}}\exp\left(-\frac{z_1^2}{2(\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2)}\right).
\end{aligned}
$$

We can now see that Z_1 does indeed follow a Gaussian distribution $\sim\mathcal{N}(0,\alpha^2\sigma_{Y_1}^2+\beta^2\sigma_{Y_2}^2)$. In hindsight, observe that this distribution respects the laws for adding any two independent random variables, namely $\mathbb{E}[\alpha Y_1+\beta Y_2]=\alpha\,\mathbb{E}[Y_1]+\beta\,\mathbb{E}[Y_2]$ and $\mathrm{var}(\alpha Y_1+\beta Y_2)=\alpha^2\,\mathrm{var}(Y_1)+\beta^2\,\mathrm{var}(Y_2)$.

Remark: Another popular solution is to use moment-generating functions, which leads to the same result.

(c) When working with joint distributions, it is convenient to use vector notation. To be precise, let us define

$$\begin{aligned}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} &= \begin{bmatrix} \alpha Y_1 + \beta Y_2 \\ \gamma Y_1 + \delta Y_2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.\end{aligned}$$

Since we do know the statistics of Y_1 and Y_2, we can quite easily find the mean and covariance by simple linear algebra.

$$\begin{aligned}\mathbb{E}\left[\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\right] &= \mathbb{E}\left[\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right] \\ &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mathbb{E}\left[\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right] \\ &= \mathbf{0}.\end{aligned}$$

For the covariance we use that Y_1 and Y_2 are independent and easily find

$$\begin{aligned}\mathbb{E}\left[\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \end{bmatrix}\right] &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mathbb{E}\left[\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}\right] \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \sigma_{Y_1}^2 & 0 \\ 0 & \sigma_{Y_2}^2 \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \\ &= \begin{bmatrix} \alpha^2\sigma_{Y_1}^2 + \beta^2\sigma_{Y_2}^2 & \alpha\gamma\sigma_{Y_1}^2 + \beta\delta\sigma_{Y_2}^2 \\ \alpha\gamma\sigma_{Y_1}^2 + \beta\delta\sigma_{Y_2}^2 & \gamma^2\sigma_{Y_1}^2 + \delta^2\sigma_{Y_2}^2 \end{bmatrix}.\end{aligned}$$

If $\alpha = \gamma$ and $\beta = \delta$, all entries in the covariance matrix are the same. In other words, Z_1 and Z_2 are maximally correlated. For independence, on the other hand, the covariance matrix must be diagonal. Thus, the condition for independence is $\alpha\gamma\sigma_{Y_1}^2 + \beta\delta\sigma_{Y_2}^2 = 0$.

(d) Since each covariance matrix is real symmetric, it has an eigenvalue decomposition as follows:

$$C = Q\Lambda Q^T.$$

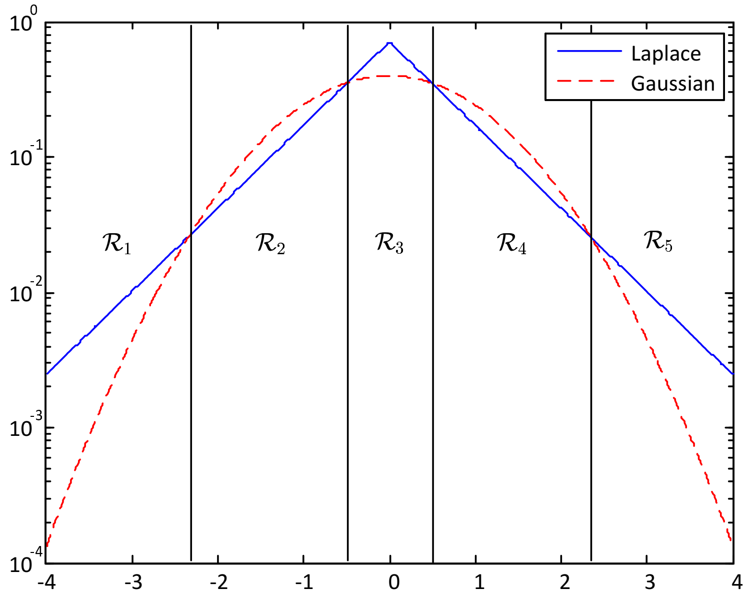
Here, Q consists of the orthonormal eigenvectors of C and Λ is a diagonal matrix of the corresponding eigenvalues. If we turn this transformation around as

$$Q^T C Q = \Lambda,$$

we observe that we can thus diagonalize any real symmetric matrix by multiplying it with its eigenvectors. Therefore, a matrix A whose *rows* are equal to the eigenvectors of C will give us a new pair of Gaussians with a diagonal covariance matrix, which means they are independent. The resulting statistics $(\sigma_{Y_1}^2, \sigma_{Y_2}^2)$ will then simply be the eigenvalues of C.

SOLUTION 2.

(a)



(b) First, we have $P_{Y|H}(y|1) = \frac{1}{\sqrt{2}} \exp\left(-\sqrt{2}|y|\right)$ (Laplace) and $P_{Y|H}(y|2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$ (Gaussian). The figure above shows the two distributions and we can divide the real line into five regions. Clearly, $\hat{H}_{ML} = 1$ if $y \in \mathcal{R}_1 \cup \mathcal{R}_3 \cup \mathcal{R}_5$ and $\hat{H}_{ML} = 2$ otherwise. To be precise,

$$
\begin{aligned}
\hat{H}_{ML}(Y = y) = \arg\max_{h \in \{1,2\}} p(y|h) &= \begin{cases} 1 & \text{if } p(y|1) \geq p(y|2) \\ 2 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } |y|^2 - 2\sqrt{2}|y| + \ln \pi \geq 0 \\ 2 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \left||y| - \sqrt{2}\right| \geq \sqrt{2 - \ln \pi} \\ 2 & \text{otherwise} \end{cases}
\end{aligned}
$$

and $a = \sqrt{2}$, $b = \sqrt{2 - \ln \pi}$.

(c) The error probability is

$$
\begin{aligned}
P_e &= P_H(1) \int_{\left||y|-\sqrt{2}\right| < \sqrt{2-\ln \pi}} P_{Y|H}(y|1)\, dy + P_H(2) \int_{\left||y|-\sqrt{2}\right| \geq \sqrt{2-\ln \pi}} P_{Y|H}(y|2)\, dy \\
&= e^{-2} \sinh\left(\sqrt{4 - 2\ln \pi}\right) + \frac{1}{2} - Q\left(\sqrt{2} - \sqrt{2 - \ln \pi}\right) + Q\left(\sqrt{2} + \sqrt{2 - \ln \pi}\right) \approx 0.43.
\end{aligned}
$$

Solution 3. We define

$$C(x_i) = 2\sigma^2 \log \Pr\{X = x_i\}$$

It is easy to show that for the optimal decision maker (MAP) in Gaussian noise, the detector finds x_i so that

$$\langle x_i, x_i\rangle - 2\langle y, x_i\rangle - C(x_i)$$

is minimized.

We know the following for any $j \neq i$

$$\langle x_i, x_i\rangle - 2\langle y_1, x_i\rangle - C(x_i) \le \langle x_j, x_j\rangle - 2\langle y_1, x_j\rangle - C(x_j) \tag{1}$$

$$\langle x_i, x_i\rangle - 2\langle y_2, x_i\rangle - C(x_i) \le \langle x_j, x_j\rangle - 2\langle y_2, x_j\rangle - C(x_j) \tag{2}$$

Now let us consider the following,

$$\begin{aligned}\langle x_i, x_i\rangle - 2\langle \alpha y_1 + (1-\alpha)y_2, x_i\rangle - C(x_i) &= \langle x_i, x_i\rangle - 2\alpha\langle y_1, x_i\rangle \\ &\quad - 2(1-\alpha)\langle y_2, x_i\rangle - C(x_i) \\ &= \alpha[\langle x_i, x_i\rangle - 2\langle y_1, x_i\rangle - C(x_i)] + \\ &\quad (1-\alpha)[\langle x_i, x_i\rangle - 2\langle y_2, x_i\rangle - C(x_i)] \\ &\le \alpha[\langle x_j, x_j\rangle - 2\langle y_1, x_j\rangle - C(x_j)] + \\ &\quad (1-\alpha)[\langle x_j, x_j\rangle - 2\langle y_2, x_j\rangle - C(x_j)]\end{aligned}$$

In the last step we used (1) and (2). We conclude

$$\langle x_i, x_i\rangle - 2\langle \alpha y_1 + (1-\alpha)y_2, x_i\rangle - C(x_i) \le \langle x_j, x_j\rangle - 2\langle \alpha y_1 + (1-\alpha)y_2, x_j\rangle - C(x_j)$$

for all $j \neq i$. Therefore, the decoder decodes $\alpha y_1 + (1-\alpha)y_2$ as x_i.

Solution 4.

(a) Given the observation (y_1, y_2), the maximum likelihood receiver computes for each hypothesis x

$$\text{score}(x) = P_{Y_1Y_2|X}(y_1, y_2|x) = P_{Y_1|X}(y_1|x)P_{Y_2|Y_1X}(y_2|y_1, x)$$

and chooses the x with the highest score. If $P_{Y_2|Y_1,X}(y_2|y_1, x) = P_{Y_2|Y_1}(y_2|y_1)$, then

$$\text{score}(x) = P_{Y_1|X}(y_1|x)P_{Y_2|Y_1}(y_2|y_1)$$

Since the factor $P_{Y_2|Y_1}(y_2|y_1)$ is common to the score of each x, the ranking of the x's will not change if it is based on the modified score

$$\text{score}'(x) = P_{Y_1|X}(y_1|x).$$

As score$'$ can be computed from y_1 alone, the receiver does not need y_2 to make its decision.

(b)

(i) With $Y_1 = X + N_1$, $Y_2 = X + N_2$, $Y_3 = X + N_1 + N_2$ with independent X, N_1, N_2

$$\begin{aligned}\Pr\{Y_3 \le y_3 | Y_1 = y_1, X = x\} &= \Pr\{X + N_1 + N_2 \le y_3 | Y_1 = y_1, X = x\} \\ &= \Pr\{N_2 \le y_3 - y_1 | Y_1 = y_1, X = x\} \\ &= \Pr\{N_2 \le y_3 - y_1\} \qquad (*) \\ &= \Pr\{Y_3 \le y_3 | Y_1 = y_1\}\end{aligned}$$

where $(*)$ follows from the independence of N_2 from X and N_1. Thus $P_{Y_3|Y_1,X}(y_3|y_1, x) = P_{Y_3|Y_1}(y_3|y_1)$ and we conclude that y_3 is irrelevant given only y_1.

(ii) Given Y_1 and Y_2, the knowledge of Y_3 would let us determine X exactly as $X = Y_1 + Y_2 - Y_3$. Such exact determination is in general not possible from Y_1 and Y_2 alone, so Y_3 is not irrelevant.

Under special circumstances the pair Y_1, Y_2 may determine X exactly, and Y_3 is irrelevant. Some examples:

(1) X is a constant;

(2) $N_1 = 0$ with probability 1; or perhaps more interestingly,

(3) X takes values only in {0,1,2,3,4,5}, N_1 takes only values in even integers and N_3 is always a multiple of 3, then, from Y_1 we know (X mod 2), from Y_2 we know (X mod 3), so we can find (X mod 6) and thus determine X.

(c) The conditional cumulative distribution of Y_2,

$$\Pr\{Y_2 \le y_2 | Y_1 = y_1, X = x\} = \Pr\{N_2 \le y_2 - x\}$$

is a function that depends on the value of x. If $\Pr\{Y_2 \le y_2 | Y_1 = y_1, X = x\}$ were equal to $\Pr\{Y_2 \le y_2 | Y_1 = y_1\}$ this would not have been the case. So, Y_2 is not irrelevant.

(d) Observe that

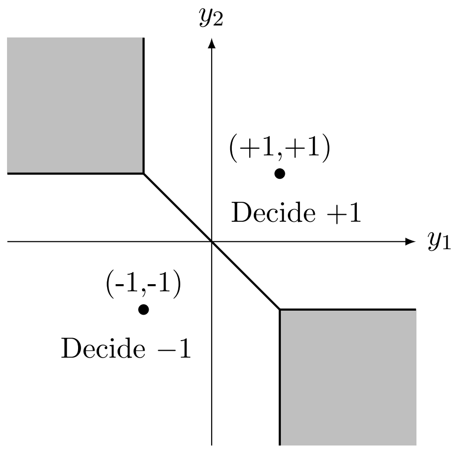
$$\log P_{Y_1 Y_2|X}(y_1, y_2|x) = \log P_{N_1}(y_1 - x) + \log P_{N_2}(y_2 - x) = -[|y_1 - x| + |y_2 - x|] - \log 2$$

Thus the optimal decision rule is

$$\begin{aligned}
\hat{H}_{\mathrm{MAP}}(y_1, y_2) &= \begin{cases} +1 & |y_1 - 1| + |y_2 - 1| < |y_1 + 1| + |y_2 + 1| \\ -1 & |y_1 - 1| + |y_2 - 1| > |y_1 + 1| + |y_2 + 1| \\ \text{either} & |y_1 - 1| + |y_2 - 1| = |y_1 + 1| + |y_2 + 1| \end{cases} \\
&= \begin{cases} +1 & g(y_1) + g(y_2) > 0 \\ -1 & g(y_1) + g(y_2) < 0 \\ \text{either} & g(y_1) + g(y_2) = 0 \end{cases}
\end{aligned}$$

with

$$\begin{aligned}
g(y) &= |y + 1| - |y - 1| \\
&= \begin{cases} -2 & y < -1 \\ 2y & -1 \le y \le 1 \\ +2 & y > 1 \end{cases}
\end{aligned}$$



The decision regions are shown in the figure with the gray zones indicating when the decision is arbitrary.

(e) Since the rule agrees with the rule derived in part (d) it is optimum for the case of equally likely messages. By symmetry, the probability of error can be computed as $\Pr\{\text{error}\} = \Pr\{\text{error}|X = -1\}$, which is the same as

$$\Pr\{Y_1 + Y_2 \ge 0 | X = -1\} = \Pr\{N_1 + N_2 \ge 2\}$$

Writing the above as

$$\int P_{N_1}(n_1)\Pr\{N_2 > 2-n_1\}\ dn_1,$$

observing that

$$\Pr\{N_2 > x\} = \begin{cases} \exp(-x)/2 & x \geq 0 \\ 1-\exp(x)/2 & x < 0, \end{cases}$$

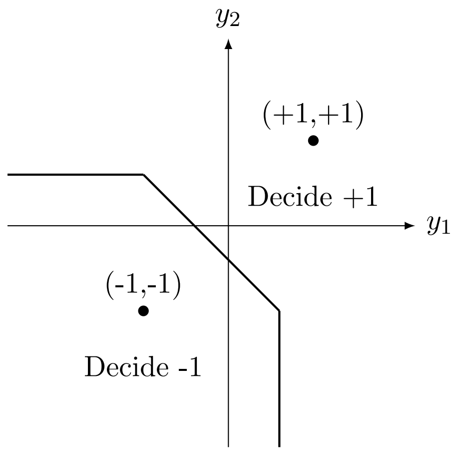
i and substituting $P_{N_1}(x)\exp(-|x|)/2$, we can compute the probability of error (above integration) as follows:

$$\int_{-\infty}^{+2} \frac{e^{-|n_1|}}{2}\frac{e^{-2+n_1}}{2}\ dn_1 + \int_{+2}^{+\infty} \frac{e^{-|n_1|}}{2}\left(1-\frac{e^{2-n_1}}{2}\right)\ dn_1 = \frac{1}{e^2}$$

(f) The MAP rule is given by decision $= \arg\max_{x\in\{+1,-1\}} P_{Y_1,Y_2|X}(y_1,y_2|x)\Pr\{X = x\}$, which, with $q = \Pr\{X = +1\}$, simplifies to

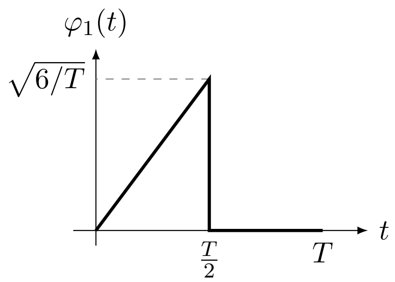
$$\hat{H}_{\text{MAP}}(y_1,y_2) = \begin{cases} +1 & g(y_1)+g(y_2) > \log((1-q)/q) \\ -1 & g(y_1)+g(y_2) < \log((1-q)/q) \\ \text{either} & g(y_1)+g(y_2) = \log((1-q)/q) \end{cases}$$

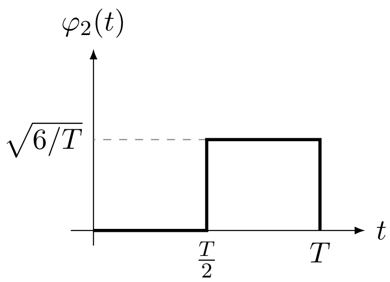
With $q > 1/2$, this has the effect of eliminating the gray zone, and shrinking the decision region for $X = -1$ as shown.



Solution 5.

(a) The following figure shows an orthonormal basis.





The different normalization ensures that their energies integrate to 1, namely

$$\begin{aligned}\int_0^T \phi_1(t)^2\,dt &= \int_0^{T/2} \left(\frac{1}{\frac{T}{2}}\sqrt{\frac{6}{T}}\cdot t\right)^2 dt\\ &= \frac{24}{T^3}\int_0^{T/2} t^2 dt\\ &= \frac{24}{T^3}\left[\frac{1}{3}t^3\right]_0^{\frac{T}{2}} = 1,\end{aligned}$$

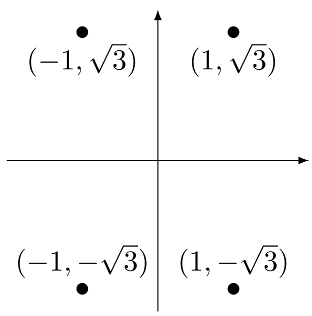
and

$$\begin{aligned}\int_0^T \phi_2(t)^2\,dt &= \int_{T/2}^T \frac{2}{T}dt\\ &= \frac{2}{T}\left(T-\frac{T}{2}\right) = 1.\end{aligned}$$

(b) For $k \in \{1,2,3,4\}$, using the orthonormal basis in (a), the signal point $\boldsymbol{s}_k$ is given by $\left(\int_{-\infty}^{\infty} s_k(t)\varphi_1(t)\,dt, \int_{-\infty}^{\infty} s_k(t)\varphi_2(t)\,dt\right)$. Thus, if $b = \sqrt{6/T}$, we must have

$$\boldsymbol{s}_1 = (1,\sqrt{3}), \boldsymbol{s}_2 = (-1,\sqrt{3}), \boldsymbol{s}_3 = (1,-\sqrt{3}), \boldsymbol{s}_4 = (-1,-\sqrt{3}),$$

and the signal space looks like



(c) The energy of each waveform is 4. For the first waveform, we can calculate:

$$\mathcal{E} \;=\; \int_{-\infty}^{\infty} s_1^2(t)\,dt = 4.$$

Note that, because our basis is orthonormal, we can equivalently calculate the energy as

$$\mathcal{E} \;=\; \|\boldsymbol{s}_1\|^2 = 1^2 + (\sqrt{3})^2 = 4.$$