École Polytechnique Fédérale de Lausanne
School of Computer and Communication Sciences

Handout 29 — Advanced Digital Communications
Solutions to Homework 12 — Dec. 19, 2016

Solution 1.

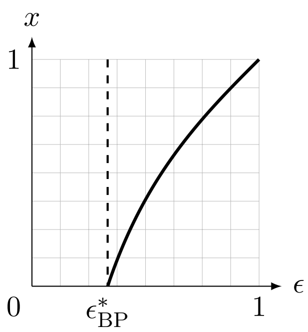
(a) The density evolution function for the (l, r)-regular ensemble is

$$f(x, \epsilon) = \epsilon \left(1 - (1-x)^{r-1}\right)^{l-1}$$

As we have seen in the class, we can plot the fixed points using the parametric expression

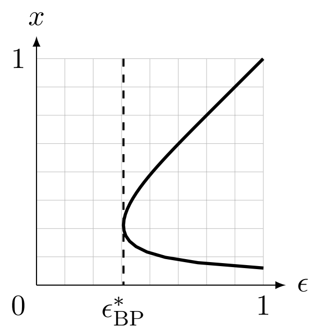
$$\epsilon(x) = \frac{x}{f(x)} = \frac{x}{\left(1-(1-x)^{r-1}\right)^{l-1}}$$

$(2, 4)$**-regular ensemble**



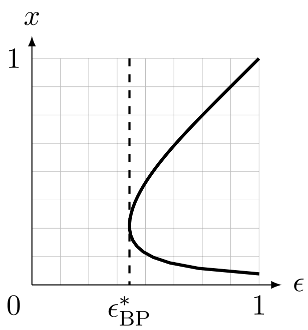
$$\epsilon^*_{\text{BP}} = \frac{1}{3}.$$

$(4, 8)$**-regular ensemble**



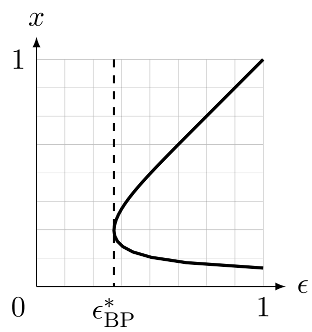
$$\epsilon^*_{\text{BP}} \approx 0.3834.$$

$(3, 6)$**-regular ensemble**



$$\epsilon^*_{\text{BP}} \approx 0.42944.$$

$(5, 10)$**-regular ensemble**



$$\epsilon^*_{\text{BP}} \approx 0.3416.$$

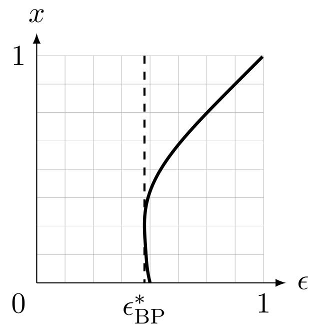
(b)

(i) The design rate of the code is given by

$$R = 1 - \frac{\int_0^1 \rho(x)\ dx}{\int_0^1 \lambda(x)\ dx} = \frac{177}{352} \approx 0.503$$

(ii) Here is the plot of fixed points obtained by plotting

$$\epsilon(x) = \frac{x}{\lambda\left(1 - \rho(1-x)\right)}$$



which shows the threshold is

$$\epsilon^*_{\text{BP}} \approx 0.47524$$

We can see that this is indeed higher than the best threshold for the previous regular ensembles (which was ≈ 0.429 for the $(3,6)$-regular ensemble).

Solution 2.

(a) Recall that if W is a BEC with erasure probability ϵ, W^s, $s \in \{-,+\}$ is BEC with erasure probability $T_s(\epsilon)$ where

$$T_-(\epsilon) = 2\epsilon - \epsilon^2 \tag{1}$$
$$T_+(\epsilon) = \epsilon^2 \tag{2}$$

Since both of the above functions are increasing in ϵ, $\epsilon_1 \leq \epsilon_2$ implies

$$T_s(\epsilon_1) \leq T_s(\epsilon_2) \qquad \forall s \in \{-,+\}$$

which proves the claim.

(b) The claim follows by induction. Fix a sign sequence $(s_1, \ldots, s_n)$ and define two sequences a_ℓ and b_ℓ as

$$a_0 = \epsilon_1 \qquad\qquad b_0 = \epsilon_2$$
$$a_\ell = T_{s_\ell}(a_{\ell-1}) \qquad\qquad b_\ell = T_{s_\ell}(b_{\ell-1})$$

Obviously a_n (respectively b_n) is the erasure probability of $W_1^{s_1\cdots s_n}$ (resp. $W_2^{s_1\cdots s_n}$). Moreover $a_0 \leq b_0$ by assumption and $a_{\ell-1} \leq b_{\ell-1}$ implies $a_\ell \leq b_\ell$ due to (a). Thus, $a_n \leq b_n$ which means $W_1^{s_1\cdots s_n} \succeq W_2^{s_1\cdots s_n}$.

(c) While the polar code that is being used for communication over the better channel (with erasure probability $\epsilon' < \epsilon$) is not the optimal one, we know from our analysis that it performs at least as well as what we would have expected if the channel indeed had erasure probability ϵ (and perhaps even better because we have strict inequality and T_- and T_+ are strictly increasing).

Solution 3. Let us focus on the *worst* channel. Fix any sign sequence $s_1, \dots, s_n$ with m minuses and $n-m$ pluses and assume this sequence contains at least one $(+,-)$ pair—i.e., a plus followed by a minus. We claim that by swapping those two signs we get a worst channel. (Hence, the worst channel is indeed the one indexed by a sequence that contains no $(+,-)$ pair — which is exactly the one claimed to index the worst channel.) Suppose $t_1, \dots, t_n$ is the sequence obtained by changing one of $(+,-)$ pairs in $s_1, \dots, s_n$ to $(-,+)$. More precisely, assume $s_j = +$ and $s_{j+1} = -$ and we let

$$\begin{aligned} t_\ell &= s_\ell, \qquad \ell = 1, 2, \dots, j-1, j+2, \dots, n \\ t_j &= s_{j+1} = - \\ t_{j+1} &= s_j = +. \end{aligned}$$

(It is obvious that the t sequence also has m minuses and $n-m$ pluses.)

Define two sequences a_ℓ, b_ℓ as

$$\begin{aligned} a_0 &= b_0 = \epsilon \\ a_\ell &= T_{s_\ell}(a_{\ell-1}) \\ b_\ell &= T_{t_\ell}(b_{\ell-1}) \end{aligned}$$

(where T_- and T_+ are defined in Equations (1) and (2) previously). Obviously, a_n is the erasure probability of $W^{s_1 \dots s_n}$ and b_n is the erasure probability of $W^{t_1 \dots t_n}$.

By construction, we, moreover, have that

$$a_\ell = b_\ell, \qquad \ell = 1, \dots, \ell - 1$$

Now,

$$a_j = T_+(a_{j-1}) = a_{j-1}^2, \qquad a_{j+1} = T_-(a_j) = 2a_j - a_j^2 = 2a_{j-1}^2 - a_{j-1}^4$$

while

$$b_j = T_-(b_{j-1}) = 2b_{j-1} - b_{j-1}^2, \qquad b_{j+1} = T_+(b_j) = b_j^2 = \left(2b_{j-1} - b_{j-1}^2\right)^2 = \left(2a_{j-1} - a_{j-1}^2\right)^2$$

It is easy to check that

$$(2\alpha^2 - \alpha^4) \leq \left(2\alpha - \alpha^2\right)^2, \qquad \forall \alpha \in [0,1]$$

thus,

$$a_{j+1} \leq b_{j+1}$$

Moreover since $s_\ell = t_\ell$ for $\ell = j+2, \dots, n$, the monotonicity of T_- and T_+ implies

$$a_\ell \leq b_\ell, \qquad \ell = j+2, \dots, n.$$

which, in particular, shows $a_n \leq b_n$. Thus, $W^{s_1 \dots s_n} \succeq W^{t_1 \dots t_n}$.

The claim on the *best* channels follows exactly in the same way.

Solution 4.

(a) If any of the synthetic channels erase, a block-error event happens. Therefore, the block-error probability of the polar code which uses channels with indices in $\mathcal{A}_n$ is lower-bounded by the erasure probability of *any* of those channels — in particular the one with the highest erasure probability.

(b)

(i) Let

$$L_-(x) := x$$
$$L_+(x) := x^2.$$

Then,

$$a_\ell := T_{s_\ell}(a_{\ell-1})$$
$$b_\ell := L_{s_\ell}(b_{\ell-1})$$

where T_- and T_+ are defined in (1) and (2) respectively. Similarly to $T_\pm$, both transformations $L_\pm$ are monotone. Moreover, $L_+(x) = T_+(x)$ while $L_-(x) \leq T_-(x)$. Therefore we can prove the claim by induction as follows: Clearly $a_0 \geq b_0$ (in fact here we have equality). Furthermore if $a_\ell \geq b_\ell$, then,

$$a_{\ell+1} = T_{s_{\ell+1}}(a_\ell) \geq L_{s_{\ell+1}}(a_\ell) \geq L_{s_{\ell+1}}(b_\ell) = b_{\ell+1}.$$

(ii) This is obvious by construction:

$$\log(b_\ell) = 2^{\mathbb{1}\{s_\ell=+\}} \log(b_{\ell-1}) = 2^{\sum_{\ell=1}^{n} \mathbb{1}\{s_\ell=+\}} \log(b_0)$$

which shows

$$\log(b_\ell) = 2^p \log(\epsilon) \quad \iff \quad b_\ell = \epsilon^{(2^p)}$$

(iii) Since $b_n \leq a_n$, if $a_n \leq 2^{-2^{\beta n}}$, then we must have

$$\epsilon^{(2^p)} \leq 2^{-2^{\beta n}} \quad \iff \quad 2^p \log(\epsilon) \leq -2^{\beta n}$$

Since $\log(\epsilon) < 0$ (as $\epsilon < 1$) the above implies

$$2^p \geq \frac{2^{\beta n}}{\log(1/\epsilon)} \quad \iff \quad p \geq \beta n - \log(\log(1/\epsilon))$$

(c)

(i) The expansion of $[z + (1-z)]^n$ is

$$1 = [z + (1-z)]^n = \sum_{i=0}^{n} \binom{n}{i} z^i (1-z)^{n-i}$$

Since all the terms in the summation are positive it can be lower-bounded by keeping any of them, say the jth term:

$$1 \geq \binom{n}{j} z^j (1-z)^{n-j}$$

which proves the claim.

(ii) Taking $z = j/n$ in (i) we have

$$\binom{n}{j} \leq \left(\frac{j}{n}\right)^{-j} \left(\frac{n-j}{n}\right)^{-(n-j)} = 2^{-n\left[\frac{j}{n}\log\left(\frac{j}{n}\right) + \frac{n-j}{n}\log\left(\frac{n-j}{n}\right)\right]} = 2^{n h_2(j/n)}$$

(iii) Since $p \geq n/2$,

$$\sum_{j=p}^{n} \binom{n}{j} \leq \sum_{j=p}^{n} \binom{n}{p} \leq (n-p+1)\binom{n}{p} \leq n\binom{n}{p} \leq n2^{nh_2(p/n)}$$

(iv) Using (iii), the fraction of such sequences is upper-bounded by

$$\frac{n2^{nh_2(p/n)}}{2^n} \leq n2^{-n[1-h_2(\beta-c(\epsilon)/n)]}$$

where the inequality follows by lower-bounding $p/n \geq \beta - c(\epsilon)/n$. Since $\beta > \frac{1}{2}$ for large enough n, $\beta - c(\epsilon)/n \geq \frac{1}{2}$, which shows the exponent of the right-hand-side of the above is strictly positive (since $h_2(\beta - c(\epsilon)/n) < 1$). Therefore, the right-hand-side of the above goes to 0 exponentially fast as $n \to \infty$.

(d) In (b) we showed that for a synthetic channel to have an erasure probability decaying faster than 2^{-N^β}, its 'index' must have at least βn pluses. In particular if $\beta > \frac{1}{2}$, its index must more than $\frac{1}{2}n$ pluses. As we showed in (c), the fraction of such sign sequences vanishes as $n \to \infty$.

(e) If the block-error probability were to decay faster than $2^{-\sqrt{N}}$, that is 2^{-N^β}, $\beta > \frac{1}{2}$, then because of (a) the erasure probability of all synthetic channels used for the transmission of information bits would have to decay as fast as 2^{-N^β}, $\beta > \frac{1}{2}$. This means $\mathcal{A}_n$ must be a subset of a vanishing fraction of channels because of (d). This requires the code rate to decay to 0.