SOLUTION 1.

(a) From the receiver's point of view, the decoding is equivalent to decoding a 4-ary signal chosen from the alphabet $\{-11\sqrt{\mathcal{E}}, -9\sqrt{\mathcal{E}}, 9\sqrt{\mathcal{E}}, 11\sqrt{\mathcal{E}}\}$. This the ML decoding rule, which is nothing but the minimum distance decoding rule will be,

$$(\hat{x}_A, \hat{x}_B) = \begin{cases} (-\sqrt{\mathcal{E}}, -\sqrt{\mathcal{E}}) & \text{if } Y \leq -10\sqrt{\mathcal{E}}, \\ (-\sqrt{\mathcal{E}}, +\sqrt{\mathcal{E}}) & \text{if } -10\sqrt{\mathcal{E}} \leq Y \leq 0, \\ (+\sqrt{\mathcal{E}}, -\sqrt{\mathcal{E}}) & \text{if } 0 \leq Y \leq 10\sqrt{\mathcal{E}}, \\ (+\sqrt{\mathcal{E}}, +\sqrt{\mathcal{E}}) & \text{if } Y \geq 10\sqrt{\mathcal{E}}. \end{cases}$$

The probability of error of the above decision rule is

$$P_{\mathrm{e}}^{\mathrm{ML}} = Q\left(\sqrt{\mathcal{E}}/\sqrt{N_0/2}\right) + \frac{1}{2}Q\left(9\sqrt{\mathcal{E}}/\sqrt{N_0/2}\right).$$

(b) We have

$$\begin{aligned} \Pr\{\text{error}|x_A = \sqrt{\mathcal{E}}, x_B = \sqrt{\mathcal{E}}\} &= \Pr\{11\sqrt{\mathcal{E}} + Z \leq 0\} \\ &= \Pr\{Z \leq -11\sqrt{\mathcal{E}}\} \\ &= Q\left(\frac{11\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right), \\ \Pr\{\text{error}|x_A = \sqrt{\mathcal{E}}, x_B = -\sqrt{\mathcal{E}}\} &= \Pr\{9\sqrt{\mathcal{E}} + Z \leq 0\} \\ &= Q\left(\frac{9\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right), \end{aligned}$$

which, together with the fact that $\Pr\{x_B = \sqrt{\mathcal{E}}\} = \Pr\{x_B = -\sqrt{\mathcal{E}}\} = \frac{1}{2}$, yield

$$\Pr\{\text{error}|x_A = \sqrt{\mathcal{E}}\} = \frac{1}{2}\left[Q\left(\frac{11\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right) + Q\left(\frac{9\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right)\right].$$

By symmetry, it follows that $\Pr\{\text{error}|x_A = \sqrt{\mathcal{E}}\} = \Pr\{\text{error}|x_A = -\sqrt{\mathcal{E}}\}$, therefore,

$$P_{\mathrm{e}}^{(A)} = \frac{1}{2}\left[Q\left(\frac{11\sqrt{2\mathcal{E}}}{\sqrt{N_0}}\right) + Q\left(\frac{9\sqrt{2\mathcal{E}}}{\sqrt{N_0}}\right)\right].$$

(c) For the genie-aide decoder the decoding problem is a simple BPSK decision whose error probability is

$$P_{\mathrm{e}}^{(B)} = Q\left(\sqrt{2\mathcal{E}/N_0}\right).$$

(d) We note that

$$\{(\hat{x}_A, \hat{x}_B) \neq (x_A, x_B)\} = \{\hat{x}_A \neq x_A\} \cup \{\hat{x}_A = x_A \text{ and } \hat{x}_B \neq x_B\},$$

that is $(\hat{x}_A, \hat{x}_B)$ can be wrong in two ways: either $\hat{x}_A$ is incorrect (and then we don't care what has happened to $\hat{x}_B$), or $\hat{x}_A$ is correct but $\hat{x}_B$ is incorrect. Now we note that

$$\{\hat{x}_A = x_A \text{ and } \hat{x}_B \neq x_B\} = \{\hat{x}_A = x_A \text{ and } \tilde{x}_B \neq x_B\}$$

because when $\hat{x}_A = x_A$, then $\tilde{Y} = Y'$, hence $\hat{x}_B = \tilde{x}_B$.

(e) Due to (d), the decision will be incorrect if either $\hat{x}_A$ or $\tilde{x}_B$ are wrong. Thus, a simple application of the union bound gives,

$$P_{\mathrm{e}} \leq P_{\mathrm{e}}^{(A)} + P_{\mathrm{e}}^{(B)}$$

On the other side we can lower-bound the error probability as

$$P_{\mathrm{e}} \geq \max\{P_{\mathrm{e}}^{(A)}, P_{\mathrm{e}}^{(B)}\}.$$

(f) Using the upper bound of (e) on P_{e} and then lower-bounding $P_{\mathrm{e}}^{\mathrm{ML}}$ by $Q(\sqrt{2\mathcal{E}/N_0})$, yields,

$$\begin{aligned}\frac{P_{\mathrm{e}}}{P_{\mathrm{e}}^{\mathrm{ML}}} \leq \frac{P_{\mathrm{e}}^{(A)} + P_{\mathrm{e}}^{(B)}}{P_{\mathrm{e}}^{\mathrm{ML}}} &\leq 1 + \frac{P_{\mathrm{e}}^{(A)}}{Q(\sqrt{2\mathcal{E}/N_0})} \\ &= 1 + \left\{\frac{Q(11\sqrt{2\mathcal{E}}/\sqrt{N_0})}{Q(\sqrt{2\mathcal{E}/N_0})} + \frac{Q(9\sqrt{2\mathcal{E}}/\sqrt{N_0})}{Q(\sqrt{2\mathcal{E}/N_0})}\right\}\end{aligned}$$

As the *hint* suggests the term inside the curly brackets in the above goes to 0 as $\mathcal{E} \to \infty$, thus,

$$\lim_{\mathcal{E}\to\infty} \frac{P_{\mathrm{e}}}{P_{\mathrm{e}}^{\mathrm{ML}}} \leq 1.$$

On the other hand, since for *any* detector $P_{\mathrm{e}} \geq P_{\mathrm{e}}^{\mathrm{ML}}$, we must have

$$\lim_{\mathcal{E}\to\infty} \frac{P_{\mathrm{e}}}{P_{\mathrm{e}}^{\mathrm{ML}}} \geq 1,$$

which proves

$$\lim_{\mathcal{E}\to\infty} \frac{P_{\mathrm{e}}}{P_{\mathrm{e}}^{\mathrm{ML}}} = 1,$$

hence the *high-SNR optimality* of the decoder.

SOLUTION 2. In the solutions we drop the time index of the transmitted signal and denote it as x_s (as opposed to $x_s[1]$).

(a) In this scenario, the receiver observes the following in two time slots:

$$\begin{aligned}Y_d[1] &= H_{sd}[1]x_s + Z_d[1], \\ Y_d[2] &= hH_{sr}[1]x_s + hZ_r[1] + Z_d[2].\end{aligned}$$

Dividing $Y_d[2]$ by h, we have the sufficient statistic

$$\begin{aligned}Y_d'[1] &= H_{sd}[1]x_s + Z_d[1], \\ Y_d'[2] &= H_{sr}[1]x_s + Z_r[1] + Z_d[2]/h.\end{aligned}$$

It is easy to see that we cannot achieve a diversity order larger than two. Next, we provide two approaches to show that the diversity order is indeed two.

1) Applying the maximum ratio combining gives the sufficient statistic $Y = (|H_{sd}[1]|^2 + |H_{sr}[1]|^2)x_s + Z$, where $Z = H_{sd}[1]^* Z_d[1] + H_{sr}[1]^*(Z_r[1] + Z_d[2]/h)$. For simplicity, we assume binary antipodal modulation of energy $\mathcal{E}_c$. Then, conditioning on $H_{sd}[1] = h_d$ and $H_{sr}[1] = h_r$, the error probability is

$$P_{\mathrm{e}}(h_d, h_r) = Q\left(\sqrt{\frac{2(|h_d|^2 + |h_r|^2)^2\mathcal{E}_c}{|h_d|^2 N_0 + |h_r|^2(N_0 + N_0/|h|^2)}}\right)$$

It is challenging to find the exact average error probability, so we consider the following upper bound on $P_e(h_d, h_r)$:

$$\begin{aligned} P_e(h_d, h_r) &\leq Q\left(\sqrt{\frac{2(|h_d|^2 + |h_r|^2)^2\mathcal{E}_c}{|h_d|^2 \alpha N_0 + |h_r|^2 \alpha N_0}}\right) \\ &= Q\left(\sqrt{\frac{2(|h_d|^2 + |h_r|^2)\mathcal{E}_c}{\alpha N_0}}\right), \end{aligned}$$

where $\alpha = 1 + 1/|h|^2$. Applying (5.46) in Lecture Notes, we have

$$\begin{aligned} P_e(\overline{\gamma}) &\leq \left(\frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}/\alpha}{2 + \overline{\gamma}/\alpha}}\right)\right)^2 \left(2 + \sqrt{\frac{\overline{\gamma}/\alpha}{2 + \overline{\gamma}/\alpha}}\right) \\ &\leq 3\left(\frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}/\alpha}{2 + \overline{\gamma}/\alpha}}\right)\right)^2 \\ &\overset{(*)}{\leq} 3\left(\frac{1}{2 + \overline{\gamma}/\alpha}\right)^2 \leq 3\alpha^2\left(\frac{1}{\overline{\gamma}}\right)^2, \end{aligned}$$

where $(*)$ follows since $0 \leq 1 - \sqrt{x} \leq 1 - x$ for all $x \in [0, 1]$. Thus, we conclude that the diversity order of the simple amplify-and-forward is two.

2) Since the concept of diversity captures only the high-SNR behavior, we use this fact to provide a simpler proof, where the idea is already used implicitly in Approach 1. Consider the following *worse* channel:

$$\begin{aligned} Y_d''[1] &= H_{sd}[1]x_s + Z_d'[1] \\ Y_d''[2] &= H_{sr}[1]x_s + Z_d'[2], \end{aligned}$$

where $\{Z_d'[n]\}$ are i.i.d. complex Gaussian with variance $(1+1/|h|^2)N_0$. This channel has diversity order two but clearly has higher error probability than the original channel. Thus, we conclude that we can also achieve diversity order two in the original channel.

(b) In this scenario, the receiver observes the following in two time slots:

$$\begin{aligned} Y_d[1] &= H_{sd}[1]x_s + Z_d[1], \\ Y_d[2] &= H_{rd}[2]H_{sr}[1]x_s + H_{rd}[2]Z_r[1] + Z_d[2]. \end{aligned}$$

Applying the maximum ratio combining gives the sufficient statistic

$$Y = (|H_{sd}[1]|^2 + |H_{sr}[1]|^2|H_{rd}[2]|^2)x_s + Z,$$

where $Z = H_{sd}[1]^* Z_d[1] + (H_{rd}[2]H_{sr}[1])^*(H_{rd}[2]Z_r[1] + Z_d[2])$. Note that Z itself is not complex Gaussian, but Z conditioned on $H_{sd}[1], H_{sr}[1], H_{rd}[2]$ is.

For simplicity, we assume binary antipodal modulation of energy $\mathcal{E}_c$. Then, conditioning on $|H_{sd}[1]|^2 = u_1$, $|H_{sr}[1]|^2 = u_2$ and $|H_{rd}[2]|^2 = u_3$, the error probability is given by

$$P_{\mathrm{e}} \;=\; Q\left(\sqrt{\frac{(u_1 + u_2u_3)^2\mathcal{E}_c}{(u_1 + u_2u_3^2 + u_2u_3)N_0/2}}\right).$$

Averaging this over exponentially distributed $|H_{sd}[1]|^2$, $|H_{sr}[1]|^2$ and $|H_{rd}[2]|^2$ gives the average error probability

$$\overline{P_{\mathrm{e}}} \;=\; \int_0^\infty\int_0^\infty\int_0^\infty Q\left(\sqrt{\frac{(u_1 + u_2u_3)^2\mathcal{E}_c}{(u_1 + u_2u_3^2 + u_2u_3)N_0/2}}\right) e^{-u_1}e^{-u_2}e^{-u_3}\,du_1du_2du_3.$$

There are many ways to deal with this integral. The one thing to remember is that we are primarily interested in an *upper bound* to $\overline{P_{\mathrm{e}}}$ which helps us characterize the diversity order. For starters, let us upper bound the error probability P_{e} as

$$\begin{aligned}
P_e &= Q\left(\sqrt{\frac{(u_1 + u_2u_3)^2\mathcal{E}_c}{(u_1 + u_2u_3^2 + u_2u_3)N_0/2}}\right)\\
&\le Q\left(\sqrt{\frac{(u_1 + u_2u_3)^2\mathcal{E}_c}{(u_1 + u_2u_3^2 + u_2u_3 + u_1u_3)N_0/2}}\right)\\
&= Q\left(\sqrt{\frac{2\mathcal{E}_c(u_1 + u_2u_3)}{N_0(1+u_3)}}\right)\\
&\overset{(*)}{\le} e^{-\frac{\mathcal{E}_c(u_1+u_2u_3)}{N_0(1+u_3)}},
\end{aligned}$$

where $(*)$ follows since $Q(x) \le e^{-x^2/2}$ for all $x \ge 0$. Hence, we can bound:

$$\begin{aligned}
\overline{P_e} &\le \int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{\mathcal{E}_c(u_1+u_2u_3)}{N_0(1+u_3)}} e^{-u_1}e^{-u_2}e^{-u_3}\,du_1du_2du_3\\
&\le \int_0^\infty \left(\int_0^\infty e^{-u_1\left(\frac{\mathcal{E}_c/N_0}{1+u_3}+1\right)}\,du_1\right)\left(\int_0^\infty e^{-u_2\left(\frac{u_3\mathcal{E}_c/N_0}{1+u_3}+1\right)}du_2\right)e^{-u_3}\,du_3\\
&= \int_0^\infty \frac{1+u_3}{1+u_3+\mathcal{E}_c/N_0}\,\frac{1+u_3}{1+u_3+u_3\mathcal{E}_c/N_0}e^{-u_3}\,du_3
\end{aligned}$$

We now cut the integral into two terms

$$\begin{aligned}
\overline{P_e} &= \int_0^1 \frac{1+u_3}{1+u_3+\mathcal{E}_c/N_0}\,\frac{1+u_3}{1+u_3+u_3\mathcal{E}_c/N_0}e^{-u_3}\,du_3\\
&\quad+\int_1^\infty \frac{1+u_3}{1+u_3+\mathcal{E}_c/N_0}\,\frac{1+u_3}{1+u_3+u_3\mathcal{E}_c/N_0}e^{-u_3}\,du_3
\end{aligned}$$

and tackle them separately.

Let us start by tackling the first term.

$$\int_0^1 \frac{1+u_3}{1+u_3+\mathcal{E}_c/N_0}\frac{1+u_3}{1+u_3+u_3\mathcal{E}_c/N_0}e^{-u_3}\,du_3$$
$$\leq \int_0^1 \frac{1+1}{1+0+\mathcal{E}_c/N_0}\frac{1+1}{1+0+u_3\mathcal{E}_c/N_0}e^{-0}\,du_3$$
$$= \frac{4}{1+\mathcal{E}_c/N_0}\int_0^1 \frac{1}{1+u_3\mathcal{E}_c/N_0}\,du_3 = \frac{4}{1+\mathcal{E}_c/N_0}\frac{\log(1+\mathcal{E}_c/N_0)}{\mathcal{E}_c/N_0}.$$

The second term can be tackled as follows:

$$\int_1^\infty \frac{1+u_3}{1+u_3+\mathcal{E}_c/N_0}\frac{1+u_3}{1+u_3+u_3\mathcal{E}_c/N_0}e^{-u_3}\,du_3$$
$$\leq \int_1^\infty \frac{1+u_3}{1+1+\mathcal{E}_c/N_0}\frac{1+u_3}{1+1+\mathcal{E}_c/N_0}e^{-u_3}\,du_3$$
$$= \frac{1}{(2+\mathcal{E}_c/N_0)^2}\int_1^\infty (1+u_3)^2 e^{-u_3}\,du_3 = \frac{10/e}{(2+\mathcal{E}_c/N_0)^2}. \quad (1)$$

Note that to derive the diversity order, it suffices to know that the integral in (1) is bounded.

Combining our expressions, we find

$$\overline{P_e} \;\leq\; \frac{4\log(1+\mathcal{E}_c/N_0)}{\mathcal{E}_c/N_0(1+\mathcal{E}_c/N_0)} + \frac{10/e}{(2+\mathcal{E}_c/N_0)^2}. \quad (2)$$

It is now a simple matter to conclude that the diversity order is 2. (Make sure that you understand why the logarithm term in (2) has no influence on the diversity order.)

Solution 3.

(a) Conditioned on $\boldsymbol{H} = (H_1, H_2)^T = (h_1, h_2)^T =: \boldsymbol{h}$, we know from the course that the probability of error of the receiver is

$$P_\mathrm{e}(\boldsymbol{h}) = Q\left(\frac{\|\boldsymbol{h}\|\sqrt{\mathcal{E}}}{\sigma}\right) \leq \exp\bigl(-\|\boldsymbol{h}\|^2\mathcal{E}/(2\sigma^2)\bigr).$$

Consequently, the average error probability is

$$\begin{aligned}
\overline{P_\mathrm{e}} &= \mathbb{E}_{\boldsymbol{H}}\bigl[P_\mathrm{e}(\boldsymbol{H})\bigr] \\
&\leq \mathbb{E}_{\boldsymbol{H}}\bigl[\exp\bigl(-\|\boldsymbol{H}\|^2\mathcal{E}/(2\sigma^2)\bigr)\bigr] \\
&= \mathbb{E}_{H_1,H_2}\bigl[\exp\bigl(-(|H_1|^2+|H_2|^2)\mathcal{E}/(2\sigma^2)\bigr)\bigr] \\
&\overset{(*)}{=} \bigl(\mathbb{E}_{H\sim\mathcal{CN}(0,1)}\bigl[\exp\bigl(-|H|^2\mathcal{E}/(2\sigma^2)\bigr)\bigr]\bigr)^2 \\
&= \left(\frac{1}{1+\frac{\mathcal{E}}{2\sigma^2}}\right)^2 \\
&\approx \frac{1}{(\mathcal{E}/2\sigma^2)^2}
\end{aligned}$$

where $(*)$ follows since H_1 and H_2 are independent circularly symmetric Gaussian RVs. Therefore, the error probability decays like $1/(\mathrm{SNR})^2$ and, we have diversity of order 2.

(b), (c) By the law of total expectation,

$$\overline{P_\text{e}} = \Pr\{F = 1\}\overline{P_{\text{e|F=1}}} + \Pr\{F = 0\}\overline{P_{\text{e|F=0}}}$$

where $\overline{P_{\text{e|F=1}}}$ (resp. $\overline{P_{\text{e|F=0}}}$) denotes the average error probability conditioned on $F = 1$ (resp. $F = 0$) — i.e., averaged *only* over Rayleigh fading coefficients.

From part (a) we know that

$$\overline{P_{\text{e|F=0}}} \leq \frac{1}{1 + (\mathcal{E}/2\sigma^2)}$$

and

$$\overline{P_{\text{e|F=1}}} \leq \left(\frac{1}{1 + (\mathcal{E}/2\sigma^2)^2}\right)^2$$

thus

$$\overline{P_\text{e}} = q\frac{1}{1 + (\mathcal{E}/2\sigma^2)} + (1-q)\frac{1}{1 + (\mathcal{E}/2\sigma^2)^2} \approx q\frac{1}{\mathcal{E}/2\sigma^2}.$$

We see that the annoying flag reduces the diversity order to 1.

SOLUTION 4.

(a) Conditional on $\boldsymbol{X} = \boldsymbol{x}$ and $\Theta = \theta$, $\boldsymbol{Y}$ is a Gaussian with mean $e^{j\theta}\boldsymbol{x}$ and covariance I. Thus

$$\begin{aligned} f(\boldsymbol{y}|\boldsymbol{x}, \theta) &= \text{const } \exp(-\|\boldsymbol{y} - e^{j\theta}\boldsymbol{x}\|^2) \\ &= \text{const } \exp(-\|\boldsymbol{y}\|^2)\exp(-\|\boldsymbol{x}\|^2)\exp(2\Re\{\langle \boldsymbol{y}, e^{j\theta}\boldsymbol{x}\rangle\}). \end{aligned}$$

As $\exp(-\|\boldsymbol{x}\|^2) = \exp(-\mathcal{E})$ is a constant it can be incorporated in the multiplicative constant. Furthermore, writing $\langle \boldsymbol{y}, \boldsymbol{x}\rangle|$ as a product of its magnitude and phase $|\langle \boldsymbol{y}, \boldsymbol{x}\rangle|e^{j\theta_0}$, we see that

$$\Re\{\langle \boldsymbol{y}, e^{j\theta}\boldsymbol{x}\rangle\} = |\langle \boldsymbol{y}, \boldsymbol{x}\rangle|\cos(\theta + \theta_0).$$

Thus,

$$\begin{aligned} f(\boldsymbol{y}|\boldsymbol{x}) &= \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{y}|\boldsymbol{x}, \theta)\, d\theta \\ &= C\exp(-\|\boldsymbol{y}\|^2)\frac{1}{2\pi}\int_0^{2\pi} \exp(2|\langle \boldsymbol{y}, \boldsymbol{x}\rangle|\cos(\theta + \theta_0))\, d\theta \\ &= C\exp(-\|\boldsymbol{y}\|^2)\frac{1}{2\pi}\int_0^{2\pi} \exp(2|\langle \boldsymbol{y}, \boldsymbol{x}\rangle|\cos(\theta))\, d\theta \end{aligned}$$

where the last step is since $\cos(\cdot)$ is periodic.

As we have seen that the likelihood ratio can be computed from the two quantities $|\langle \boldsymbol{y}, \boldsymbol{x}_0\rangle|$ and $|\langle \boldsymbol{y}, \boldsymbol{x}_1\rangle|$ the maximum likelihood decoder can base its decision only on these two positive real numbers.

(b)

$$\begin{aligned}
\pi g'(a) &= \int_0^{2\pi} \cos(\theta) e^{2a\cos(\theta)}\, d\theta \\
&= \int_0^{\pi} \cos(\theta) e^{2a\cos(\theta)}\, d\theta + \int_{\pi}^{2\pi} \cos(\theta) e^{2a\cos(\theta)}\, d\theta \\
&= \int_0^{\pi} \cos(\theta)\left[e^{2a\cos(\theta)} - e^{-2a\cos(\theta)}\right] d\theta \\
&\geq 0
\end{aligned}$$

Since $\cos(\theta) \geq 0$ on $[0, \pi]$ (and thus $e^{2a\cos(\theta)} \geq 1 \geq e^{-2a\cos(\theta)}$).

Consequently $f(\boldsymbol{y}|\boldsymbol{x})$ is an increasing function of $|\langle \boldsymbol{y}, \boldsymbol{x}\rangle|$ and so the decision rule given is the maximum likelihood rule. It thus minimizes the probability of error.

(c) By symmetry it suffices to compute the error probability conditioned on the hypothesis $m = 0$. In this case

$$|\langle \boldsymbol{Y}, \boldsymbol{x}_0\rangle| = |\langle \boldsymbol{Y}, e^{j\Theta}\boldsymbol{x}_0\rangle| = |\mathcal{E} + \langle \boldsymbol{Z}, e^{j\Theta}\boldsymbol{x}_0\rangle|$$

and

$$|\langle \boldsymbol{Y}, \boldsymbol{x}_1\rangle| = |\langle \boldsymbol{Z}, \boldsymbol{x}_1\rangle|.$$

Observe now that $W_0 = \langle \boldsymbol{Z}, e^{j\Theta}\boldsymbol{x}_0\rangle$ and $W_1 = \langle \boldsymbol{Z}, \boldsymbol{x}_1\rangle$ are independent, circularly symmetric complex Gaussian random variables with $\mathbb{E}[|W_0|^2] = \mathbb{E}[|W_1|^2] = \mathcal{E}$. Also, $\Pr\{|W_1|^2 > a\} = \exp(-a/\mathcal{E})$. Thus, conditioned on W_0, the error probability is

$$\Pr\{\text{error}|W_0\} = \exp(-|\mathcal{E} + W_0|^2/\mathcal{E})$$

and

$$\Pr\{\text{error}\} = \frac{1}{\pi\mathcal{E}} \int_{\mathbb{C}} e^{-|w_0|^2/\mathcal{E}} e^{-|\mathcal{E}+w_0|^2/\mathcal{E}}\, dw_0$$

Noting that $|w_0|^2 + |\mathcal{E} + w_0|^2 = 2[|w_0 + \mathcal{E}/2|^2] + \mathcal{E}^2/2$, we see

$$\Pr\{\text{error}\} = e^{-\mathcal{E}/2} \frac{1}{\pi\mathcal{E}} \int_{\mathbb{C}} e^{-2|w_0+\mathcal{E}/2|^2/\mathcal{E}}\, dw_0.$$

As $\frac{2}{\pi\mathcal{E}}\exp(-2|w+\mathcal{E}/2|^2/\mathcal{E})$ is the density of a complex Gaussian with mean $-\mathcal{E}/2$ and variance $\mathcal{E}/2$, we see that

$$\Pr\{\text{error}\} = \frac{1}{2}e^{-\mathcal{E}/2}.$$

(d) The situation 'with H' is equivalent to the situation 'without H' but with sent signals $H\boldsymbol{x}_0$ or $H\boldsymbol{x}_1$. The optimal decoder would thus compare

$$|\langle \boldsymbol{Y}, H\boldsymbol{x}_0\rangle| = |H||\langle \boldsymbol{Y}, \boldsymbol{x}_0\rangle| \quad \text{to} \quad |\langle \boldsymbol{Y}, H\boldsymbol{x}_1\rangle| = |H||\langle \boldsymbol{Y}, \boldsymbol{x}_1\rangle|$$

which is the same as comparing

$$|\langle \boldsymbol{Y}, \boldsymbol{x}_0\rangle| \quad \text{to} \quad |\langle \boldsymbol{Y}, \boldsymbol{x}_1\rangle|.$$

Consequently, the knowledge of H does not help in deciding the message and there is no need to waste effort to estimate it.

Solution 5. Consider transmission over two independent (parallel) channels

$$Y_1 = H_1 X_1 + Z_1 \qquad \text{and} \qquad Y_2 = H_2 X_2 + Z_2$$

where Z_1 and Z_2 are independent circularly symmetric Gaussian noises of mean zero and variance N_0 and H_1 and H_2 are independent Rayleigh fading coefficients, i.e., they are circularly symmetric complex Gaussian random variables with variance 1. We want to transmit *two* bits across a single use of two channels and we would like to compare the following two transmission strategies:

Strategy A Send independent BPSK symbols with power $\mathcal{E}$ across independent channels, that is we pick $X_1 \in \{\pm\sqrt{\mathcal{E}}\}$ and $X_2 \in \{\pm\sqrt{\mathcal{E}}\}$ independently.

Strategy B Map the 2 bits to a symbol from a symmetric 4-PAM constellation with power $\mathcal{E}$ and send *the same signal through both channels.* That is we set $X_1 = X_2 \in \{-3a, -a, a, 3a\}$.

(a) The average energy of the 4-PAM constellation is $5a^2$, thus $a = \sqrt{\mathcal{E}/5}$ will lead to an average energy of $\mathcal{E}$.

(b) For **Strategy A**, a sufficient statistic for decision, using maximum ratio combining, is

$$\begin{aligned} Y_1' &= h_1^* Y_1 = |h_1|^2 X_1 + h_1^* Z_1 \\ Y_2' &= h_2^* Y_2 = |h_2|^2 X_2 + h_2^* Z_2 \end{aligned}$$

Thus, the probabilities of incorrect decoding of X_1 and X_2 are respectively,

$$P_{\mathrm{e}}^{(1)} = Q\left(\sqrt{\frac{2|h_1|^2\mathcal{E}}{N_0}}\right) \qquad \text{and} \qquad P_{\mathrm{e}}^{(2)} = Q\left(\sqrt{\frac{2|h_2|^2\mathcal{E}}{N_0}}\right)$$

(note that since X_1 and X_2 are both real-valued signals, for decision, only the real components of Z_1 and Z_2 matter whose variances are $|h_1|^2 N_0/2$ and $|h_2|^2 N_0/2$). The overall error probability is, hence,

$$\begin{aligned} P_{\mathrm{e}}^{(A)}(h_1, h_2) &= \Pr\big(\{\text{error in } X_1\} \cup \{\text{error in } X_2\}\big) \\ &= Q\left(\sqrt{2\frac{|h_1|^2\mathcal{E}}{N_0}}\right) + Q\left(\sqrt{2\frac{|h_2|^2\mathcal{E}}{N_0}}\right) - Q\left(\sqrt{2\frac{|h_1|^2\mathcal{E}}{N_0}}\right) \cdot Q\left(\sqrt{2\frac{|h_2|^2\mathcal{E}}{N_0}}\right). \end{aligned}$$

For **Strategy B** a sufficient statistic is,

$$Y' = h_1^* Y_1 + h_2^* Y_2 = (|h_1|^2 + |h_2|^2) X + (h_1^* Z_1 + h_2^* Z_2).$$

The above leads to an overall error probabilty of

$$P_{\mathrm{e}}^{(B)} = \frac{3}{2} Q\left(\sqrt{\frac{2(|h_1|^2 + |h_2|^2)\mathcal{E}}{5N_0}}\right).$$

(Note again that only real parts of noise matter.)

(c) If $h_1 = h_2 = h$, for large $\mathcal{E}$, $P_{\rm e}^{(A)}$ decay like $\exp\bigl(-|h|^2\mathcal{E}/N_0\bigr)$ while $P_{\rm e}^{(B)}$ scales like $\exp\bigl(-2|h|^2\mathcal{E}/(5N_0)\bigr)$. Therefore, **Strategy A** is preferable.

However, if $h_1 \neq h_2$, in particular, $|h_1| < \frac{1}{2}|h_2|$, then $P_{\rm e}^{(A)}$ scales like $\exp\bigl(-|h_1|^2\mathcal{E}/N_0\bigr)$ which is larger than $\exp\bigl(-(|h_1|^2 + |h_2|^2)\mathcal{E}/(5N_0)\bigr)$. Therefore, **Strategy B** would be preferable. (It is obvious that the case $|h_2| < \frac{||}{h_1}$ is treated similarly.)

(d) We know from the lectures that

$$\mathbb{E}_H\left[Q\left(\frac{2|h|^2\mathcal{E}}{N_0}\right)\right] = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right)$$

where $\overline{\gamma} = \mathcal{E}/(N_0/2)$. Consequently,

$$\overline{P_{\rm e}^{(A)}} = \left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right) - \frac{1}{4}\left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right)^2 \approx \frac{1}{\overline{\gamma}}.$$

For the other strategy, it can be checked (see Equation (5.46) of your lecture notes) that

$$\overline{P_{\rm e}^{(B)}} = \frac{3}{2}\left(\frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right)\right)^2 \sum_{\ell=0}^{1}\binom{1+\ell}{\ell}\left(\frac{1}{2}\left(1 + \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right)\right)^\ell \approx \frac{3}{2}\cdot\frac{25}{(2\overline{\gamma})^2}.$$

(e) From (d) it is clear that the **Strategy B** gives us a diversity order of 2 which is better than **Strategy A** which leads to a diversity of order 1.