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Handout 5 Advanced Digital Communications
Solutions to Homework 1 Oct. 3, 2016

Solution 1.

(a) Let us start from the definition of DTFT:

$$\begin{aligned}
X(f) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn} \\
&\overset{(*)}{=} \sum_{n=-\infty}^{\infty} x^*[-n]e^{j2\pi f(-n)} \\
&\overset{(\dagger)}{=} \sum_{m=-\infty}^{\infty} x^*[m]e^{j2\pi m} \\
&= \sum_{m=-\infty}^{\infty} \left(x[m]e^{-j2\pi fm}\right)^* \\
&= X(f)^*,
\end{aligned}$$

where $(*)$ follows from $x[n] = x^*[-n]$ and $(\dagger)$ follows by a change of variable $m = -n$. Since $X(f) = X(f)^*$, we must have that $X_{im}(f) = 0$ and $X(f)$ is real-valued.

Similarly, let us start from the definition of Z-transform:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&\overset{(*)}{=} \sum_{n=-\infty}^{\infty} x^*[-n]z^{-n} \\
&\overset{(\dagger)}{=} \sum_{m=-\infty}^{\infty} x^*[m]z^{m} \\
&\overset{(\#)}{=} \sum_{m=-\infty}^{\infty} \left(x[m]\left(z^*\right)^m\right)^* \\
&= \left(\sum_{m=-\infty}^{\infty} x[m]\left(1/z^*\right)^{-m}\right)^* \\
&= X^*\left(1/z^*\right),
\end{aligned}$$

where $(*)$ follows from $x[n] = x[-n]$, $(\dagger)$ follows by a change of variable $m = -n$ and $(\#)$ follows from $(ab)^* = a^*b^*$.

(b) The Fourier transform of $(x * y)[n]$ is given by

$$\begin{aligned}
\mathcal{F}\{x * y\}(f) &= \sum_{n=-\infty}^{\infty} (x * y)[n] e^{-j2\pi fn} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] y[n-k] e^{-j2\pi fk} e^{-j2\pi f(n-k)} \\
&= \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi fk} \left(\sum_{n=-\infty}^{\infty} y[n-k] e^{-j2\pi f(n-k)} \right) \\
&\overset{(*)}{=} \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi fk} \left(\sum_{m=-\infty-k}^{\infty-k} y[m] e^{-j2\pi fm} \right) \\
&= X(f)Y(f),
\end{aligned}$$

where $(*)$ follows by a change of variable $m = n - k$.

(c) The Fourier transform of $y[n] = (x * \bar{x})[n]$ is given by

$$\begin{aligned}
\mathcal{F}\{x * \bar{x}\}(f) &= \sum_{n=-\infty}^{\infty} (x * \bar{x})[n] e^{-j2\pi fn} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] \bar{x}[n-k] e^{-j2\pi fk} e^{-j2\pi f(n-k)} \\
&\overset{(*)}{=} \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi fk} \left(\sum_{n=-\infty}^{\infty} x^*[k-n] e^{-j2\pi f(n-k)} \right) \\
&\overset{(\dagger)}{=} \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi fk} \left(\sum_{m=-\infty-k}^{\infty-k} x^*[m] e^{j2\pi fm} \right) \\
&= \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi fk} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi fm} \right)^* \\
&= X(f)X^*(f) \\
&= |X(f)|^2,
\end{aligned}$$

where $(*)$ follows from the definition of $\bar{x}[n]$ and $(\dagger)$ follows by a change of variable $m = k - n$.

You can also use the result in (b): first you need to find out that the Fourier transform of $x^*[-n]$ is given by $X^*(f)$. Then the result in (b) shows that the Fourier transform of $y[n]$ is just the product $X(f)X^*(f) = |X(f)|^2$.

(d) The DFT of $x[n]$ can be rewritten as

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}k\times 0} & \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}k\times 1} & \cdots & \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}k(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}.$$

Thus, if we let $\left[\frac{1}{\sqrt{N}}e^{-j\frac{2\pi}{N}k\times 0} \quad \frac{1}{\sqrt{N}}e^{-j\frac{2\pi}{N}k\times 1} \quad \cdots \quad \frac{1}{\sqrt{N}}e^{-j\frac{2\pi}{N}k(N-1)}\right]$ be the $(k+1)$-th row of F, for $k = 0, 1, \cdots, N-1$, then the resulting F is the desired Fourier matrix with the $(k+1, n+1)$-entry given by

$$F[k+1, n+1] = \frac{1}{\sqrt{N}}e^{-j\frac{2\pi}{N}kn}, \quad k, n = 0, 1, \cdots, N-1.$$

Note that we also have

$$\mathbf{X} = F\mathbf{x},$$

where $\mathbf{X} = (X[0], X[1], \cdots, X[N-1])^T$ and $\mathbf{x} = (x[0], x[1], \cdots, x[N-1])^T$. Thus, the DFT is a linear transform.

To prove $F^{-1} = F^H$, we show that $FF^H = I$: (note that $AB = I$ implies $BA = I$)

$$\begin{aligned}
\sum_{\ell=0}^{N-1} F[k+1, \ell+1]F^H[\ell+1, n+1] &= \frac{1}{N}\sum_{\ell=0}^{N-1} e^{-j\frac{2\pi}{N}k\ell}e^{j\frac{2\pi}{N}\ell n} \\
&= \frac{1}{N}\sum_{\ell=0}^{N-1} e^{j\frac{2\pi}{N}\ell(n-k)} \\
&= \begin{cases} \frac{1}{N}\sum_{\ell=0}^{N-1} 1 = 1 & \text{if } k = n, \\ \frac{1}{N}\frac{1-e^{j2\pi(k-n)}}{1-e^{j\frac{2\pi}{N}(k-n)}} = 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The DFT matrices of dimension 2, 3 and 4 are the followings:

$$\begin{aligned}
F_2 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \\
F_3 &= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}, \\
F_4 &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}.
\end{aligned}$$

SOLUTION 2.

(a)

$$\begin{aligned}
\mathbb{E}[aX + bY] &= \sum_x \sum_y (ax + by)p_{XY}(x, y) \\
&= \sum_x ax \sum_y p_{XY}(x, y) + \sum_y by \sum_x p_{XY}(x, y) \\
&= a\sum_x x\, p_X(x) + b\sum_y y\, p_Y(y) \\
&= a\,\mathbb{E}[X] + b\,\mathbb{E}[Y].
\end{aligned}$$

(b) If X and Y are independent, we have $p_{XY}(x,y) = p_X(x)p_Y(y)$, then

$$\begin{aligned}
\mathbb{E}[X \cdot Y] &= \sum_x \sum_x x \cdot y \cdot p_{XY}(x,y) \\
&= \sum_x \sum_y x \cdot y \cdot p_X(x) \cdot p_Y(y) \\
&= \sum_x x \cdot p_X(x) \sum_y y \cdot p_Y(y) \\
&= \mathbb{E}[X] \cdot \mathbb{E}[Y]
\end{aligned}$$

(c) For the first example, suppose $\Pr(X = 0, Y = 1) = \Pr(X = 1, Y = 0) = \frac{1}{2}$, and $\Pr(X = 0, Y = 0) = \Pr(X = 1, Y = 1) = 0$. X, Y are dependent, and we have $\mathbb{E}[X \cdot Y] = 0$ while $\mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{4}$

For the second example, suppose $\Pr(X = -1, Y = 0) = \Pr(X = 0, Y = 1) = \Pr(X = 1, Y = 0) = \frac{1}{3}$. X, Y are dependent. Obviously we have $\mathbb{E}[X \cdot Y] = 0$, and furthermore $\mathbb{E}[X] = 0$, hence $\mathbb{E}[X]\,\mathbb{E}[Y] = 0$.

(d) If X and Y are independent, we have $p_{XY}(x,y) = p_X(x)p_Y(y)$, then

$$\begin{aligned}
\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] &= \sum_x \sum_y (x - \mathbb{E}[X])(y - \mathbb{E}[Y])\, p_{XY}(x,y) \\
&= \sum_x \sum_y (x - \mathbb{E}[X])(y - \mathbb{E}[Y])\, p_X(x)p_Y(y) \\
&= \sum_x (x - \mathbb{E}[X])\, p_X(x) \sum_y (y - \mathbb{E}[Y])\, p_Y(y) \\
&= (\mathbb{E}[X] - \mathbb{E}[X])(\mathbb{E}[Y] - \mathbb{E}[Y]) = 0.
\end{aligned}$$

Thus, X and Y are uncorrelated.

(e) First, we have

$$\begin{aligned}
\mathrm{cov}(X,Y) &= \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] \\
&= \mathbb{E}\left[XY - X\,\mathbb{E}[Y] - \mathbb{E}[X]Y + \mathbb{E}[X]\,\mathbb{E}[Y]\right] \\
&= \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y].
\end{aligned}$$

Thus, $\mathrm{cov}(X,Y) = 0$ if and only if $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Then,

$$\begin{aligned}
\sigma^2_{aX+bY} &= \mathbb{E}\left[(aX + bY - \mathbb{E}[aX + bY])^2\right] \\
&= \mathbb{E}[(aX + bY)^2] - (\mathbb{E}[aX + bY])^2 \\
&= a^2\,\mathbb{E}[X^2] + 2ab\,\mathbb{E}[X \cdot Y] + b^2\,\mathbb{E}[Y^2] - a^2\,\mathbb{E}[X]^2 - 2ab\,\mathbb{E}[X]\,\mathbb{E}[Y] - b^2\,\mathbb{E}[Y]^2 \\
&= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2) + b^2(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) \\
&= a^2\sigma_X^2 + b^2\sigma_Y^2.
\end{aligned}$$

We remark that since the independence of X and Y implies $\mathrm{cov}(X,Y) = 0$, we also have $\sigma^2_{aX+bY} = a^2\sigma_X^2 + b^2\sigma_Y^2$ if X and Y are independent.

Solution 3.

(a)

$$\begin{aligned}\sum_{n>0} \Pr\{N \geq n\} &= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \Pr\{N = m\} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{m} \Pr\{N = m\} \\ &= \sum_{m=1}^{\infty} m \Pr\{N = m\} \\ &= \mathbb{E}[N].\end{aligned}$$

(b)

$$\begin{aligned}\int_0^{\infty} \Pr\{X \geq a\}\; da &= \int_0^{\infty} \int_a^{\infty} f_X(t)\; dt\; da \\ &= \int_0^{\infty} \int_0^{t} f_X(t)\; da\; dt \\ &= \int_0^{\infty} t f_X(t)\; dt \\ &= \mathbb{E}[X].\end{aligned}$$

(c) The main point is to note that $G(t) = \Pr\{X \geq t\}$ is a non-increasing function of t. So for any fixed value of $a > 0$, the rectangle between point $(0,0)$ and $(a, G(a))$ lies below the function $G(t)$. In conclusion, it follows from the discussion above that

$$aG(a) = \int_0^a G(a)\; dt \leq \int_0^a G(t)\; dt \leq \int_0^{\infty} G(t)\; dt,$$

which, using (b), means

$$a \Pr\{X \geq a\} \leq \mathbb{E}[X].$$

(d) Define $X \geq 0$ as

$$X := (Y - \mathbb{E}[Y])^2.$$

Using Markov inequality (part (c)) we have

$$\Pr\{(Y - \mathbb{E}[Y])^2 \geq a\} = \Pr\{X \geq a\} \leq \frac{\mathbb{E}[X]}{a} = \frac{\sigma_Y^2}{a}.$$

Setting $b = \sqrt{a}$ in the above we get

$$\Pr\{|Y - \mathbb{E}[Y]| \geq b\} = \Pr\{(Y - \mathbb{E}[Y])^2 \geq a\} \leq \frac{\sigma_Y^2}{b^2}.$$

(e) Fix $s \geq 0$, then we have

$$\begin{aligned}\Pr\{Z \geq b\} &\leq \Pr\{s(Z - b) \geq 0\} \\ &= \Pr\{e^{s(Z-b)} \geq e^0\} \\ &\overset{(*)}{\leq} \mathbb{E}\big[e^{s(Z-b)}\big],\end{aligned}$$

where $(*)$ follows from the Markov inequality.

Solution 4.

(a) $\Pr\{X_1 \leq X_2\} = \frac{1}{2}$. We know because of independence we have, $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$, and we want to find the probability of x_1 being minimum of two. This event partitions the probability space into two equal subsets, the other one is x_2 being the minimum of the two. The only problem is the boundary line $x_1 = x_2$, which we assume is a part of the first sub-set, but because X is a continuous random variable the line $x_1 = x_2$ has zero probability mass and because $f_{X_1}(x_1)f_{X_2}(x_2)$ is symmetric with respect to the line $x_1 = x_2$, we conclude that the event $\min(x_1, x_2) = x_1$ partitions the whole probability space into two equally probable regions.

(b) $\Pr\{X_1 \leq X_2; X_1 \leq X_3\} = \frac{1}{3}$. We follow the exact same argument as the part (a), this time the probability space is partitioned into three equally probable subsets, in each of subsets one of the three random variable is the minimum.

(c) We know that $\Pr\{N \geq 1\} = 1$ (trivially) and for $n \geq 2$,

$$\Pr\{N \geq n\} = \Pr\{X_1 \leq X_2; X_1 \leq X_3; \dots; X_1 \leq X_{n-1}\} = \frac{1}{n-1}$$

due to the same reasoning as in parts (a) and (b).

(d) Using part (a) of Problem 3,

$$\mathbb{E}[N] = \sum_{n\geq 1} \Pr\{N \geq n\} = 1 + \sum_{n\geq 2} \Pr\{N \geq n\} = 1 + \sum_{n\geq 2} \frac{1}{n-1} = \infty$$

(since the series, $\sum_{j=1}^{n} \frac{1}{n}$ is divergent).

(e) The symmetry of $f_{X_1}(x_1)f_{X_2}(x_2)$ still holds because of independence bu in the discrete case it is possible to put some probability mass on the line $x_1 = x_2$. Therefore, in the discrete case the event $\{X_1 \leq X_2\}$ does not partition the whole probability space into two equally probable subspaces. However, we can still conclude that $\Pr\{X_1 < X_2\} = \Pr\{X_2 < X_1\}$ (because of the symmetry). On the other hand, $\Pr\{X_1 < X_2\} + \Pr\{X_1 = X_2\} + \Pr\{X_1 > X_2\} = 1$. Consequently, we conclude that $\Pr\{X_1 \leq X_2\} \geq \frac{1}{2}$. Similarly, we can conclude that

$$\Pr\{X_1 \leq X_2; X_1 \leq X_3; \dots; X_1 \leq X_{n-1}\} \geq \frac{1}{n-1}.$$

Following the steps in part (d), we can show that

$$\mathbb{E}[N] \geq 1 + \sum_{n\geq 2} \frac{1}{n-1} = \infty.$$