SOLUTION 1.

(a) Note that $\forall m, n$,

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)} = \begin{cases} 1 & \text{if } n = m \\ \frac{1}{N}\frac{1-e^{j2\pi(n-m)}}{1-e^{j\frac{2\pi}{N}(n-m)}}, & \text{if } n \neq m, \end{cases}$$
$$= \mathbb{1}\{n = m\}. \qquad (\star)$$

Consequently, $\forall m$,

$$\begin{aligned}
\mathcal{F}\{\mathcal{F}^{-1}\{X\}\}[m] &= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\mathcal{F}^{-1}\{X\}[k]e^{-j\frac{2\pi}{N}km} \\
&= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}X[n]e^{j\frac{2\pi}{N}nk}\right)e^{-j\frac{2\pi}{N}km} \\
&= \sum_{n=0}^{N-1}X[n]\left(\frac{1}{N}\sum_{k=0}^{n-1}e^{-j\frac{2\pi}{N}km}e^{j\frac{2\pi}{N}kn}\right) \\
&\overset{(\star)}{=} \sum_{n=0}^{N-1}X[n]\mathbb{1}\{n = m\} \\
&= X[m].
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{F}\{x \circledast y\}[n] &= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}(x \circledast y)[k]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}x[m]y[(k-m) \bmod N]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}x[m]e^{-j\frac{2\pi}{N}mn} \cdot y[(k-m) \bmod N]e^{-j\frac{2\pi}{N}(k-m)n} \\
&\overset{(a)}{=} \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}x[m]e^{-j\frac{2\pi}{N}mn} \cdot y[(k-m) \bmod N]e^{-j\frac{2\pi}{N}[(k-m) \bmod N]n} \\
&\overset{(b)}{=} \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}x[m]e^{-j\frac{2\pi}{N}mn} \cdot \sum_{\ell=0}^{N-1}y[\ell]e^{-j\frac{2\pi}{N}\ell n}. \\
&= \sqrt{N}X[n]Y[n].
\end{aligned}$$

In the above (a) follows since $e^{-j\frac{2\pi}{N}(k-m)} = e^{-j\frac{2\pi}{N}[(k-m) \bmod N]}$ and (b) by setting $\ell := (k-m) \bmod N$.

(c)

$$\begin{aligned}
Y[n] &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y[k] e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[(k+m) \bmod N] e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[(k+m) \bmod N] e^{-j\frac{2\pi}{N}(k+m)n} e^{j\frac{2\pi}{N}mn} \\
&\overset{(a)}{=} e^{j\frac{2\pi}{N}mn} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[(k+m) \bmod N] e^{-j\frac{2\pi}{N}[(k+m) \bmod N]n} \\
&\overset{(b)}{=} e^{j\frac{2\pi}{N}mn} \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} x[\ell] e^{-j\frac{2\pi}{N}\ell n} \\
&= e^{j\frac{2\pi}{N}mn} X[n].
\end{aligned}$$

In the above (a) follows since $e^{-j\frac{2\pi}{N}(k+m)} = e^{-j\frac{2\pi}{N}[(k+m) \bmod N]}$ and (b) by setting $\ell := (k+m) \bmod N$.

(d) Using the identity ($\star$) that we proved in part (a),

$$\begin{aligned}
\sum_{n=0}^{N-1} X^*[n]Y[n] &= \sum_{n=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}kn} \right)^* \cdot \left(\frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} y[\ell] e^{-j\frac{2\pi}{N}\ell n} \right) \\
&= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \frac{1}{N} x^*[k] y[\ell] e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}\ell n} \\
&= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} x^*[k] y[\ell] \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}\ell n} \right) \\
&\overset{(\star)}{=} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} x^*[k] y[\ell] \mathbb{1}\{\ell = k\} \\
&= \sum_{k=0}^{N-1} x^\star[k] y[k].
\end{aligned}$$

Solution 2.

(a) No worries! Using Bayes' rule, the chance that you have Dysania given the positive test outcome is

$$\begin{aligned}
\mathbb{P}\{\text{Dysania} \mid \text{test positive}\} &= \frac{\mathbb{P}\{\text{test positive} \mid \text{Dysania}\}\mathbb{P}\{\text{Dysania}\}}{\mathbb{P}\{\text{test positive}\}} \\
&= \frac{0.99 \cdot 10^{-6}}{(1 - 10^{-6}) \cdot 0.01 + 10^{-6} \cdot 0.99} \sim 10^{-4}.
\end{aligned}$$

(b) A random variable is a mapping from the space of outcomes to the reals!

(c)

$$f(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{(x+1)^2}{6}}.$$

The chance that a random variable X drawn from this distribution takes on a value larger than 4 is equal to

$$Q(5/\sqrt{3}).$$

For large x the dominant behavior of the function $Q(x)$ is

$$e^{-x^2/2}.$$

(d) $\Pr\{X \geq \alpha\} \leq \mathbb{E}[X]/\alpha$.

(e) By linearity of expectation the answer is simply the sum of the means, hence -2, no real computation required.

(f) The upper bound is the standard union bound, the lower bound follows by the exclusion-inclusion principle.

SOLUTION 3.

(a) The optimal decision rule is the MAP rule, namely,

$$\begin{aligned} \hat{X}_{\text{MAP}}(y) &= \arg\max_{x \in \mathcal{X}} \Pr\{X = x | Y = y\} \\ &= \arg\max_{x \in \mathcal{X}} f_{Y|X}(y|x) \Pr\{X = x\}, \end{aligned}$$

where $\mathcal{X}$ is the space of hypotheses (in our case $\mathcal{X} = \{x_1, x_2\}$). Since the two hypotheses $\{X = x_1\}$ and $\{X = x_2\}$ are equally probable, the MAP decision rule reduces to

$$\hat{X}_{\text{MAP}}(y) = \arg\max_{x \in \mathcal{X}} f_{Y|X}(y|x).$$

Conditioned on $\{X = x\}$, Y is a Gaussian random variable with mean x and variance σ^2, therefore, the decision rule will be

$$\begin{aligned} f_{Y|X}(y|x_1) \underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\gtrless}} f_{Y|X}(y|x_2) & \qquad \Longleftrightarrow \\ \frac{1}{\sigma\sqrt{2\pi}} \exp\Big\{-\frac{(y-x_1)^2}{2\sigma^2}\Big\} \underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\gtrless}} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big\{-\frac{(y-x_2)^2}{2\sigma^2}\Big\} & \qquad \Longleftrightarrow \\ \frac{(y-x_1)^2}{2\sigma^2} \underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\lessgtr}} \frac{(y-x_2)^2}{2\sigma^2} & \qquad \Longleftrightarrow \\ y \underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\lessgtr}} \frac{x_1+x_2}{2}. & \end{aligned}$$

(b) In this case the MAP rule is

$$
\begin{aligned}
p \cdot f_{Y|X}(y|x_1) &\underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\gtrless}} (1-p) \cdot f_{Y|X}(y|x_2) && \iff \\
p \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\Big\{-\frac{(y-x_1)^2}{2\sigma^2}\Big\} &\underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\gtrless}} (1-p) \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\Big\{-\frac{(y-x_2)^2}{2\sigma^2}\Big\} && \iff \\
\frac{(y-x_1)^2}{2\sigma^2} - \log(p) &\underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\lessgtr}} \frac{(y-x_2)^2}{2\sigma^2} - \log(1-p) && \iff \\
y &\underset{\hat{X}=x_2}{\overset{\hat{X}=x_1}{\lessgtr}} \frac{x_1+x_2}{2} + \frac{\sigma^2}{x_2-x_1} \log \frac{p}{1-p}.
\end{aligned}
$$

(As a sanity-check you can observe that as $p \to 1$ the right-hand-side of the above approaches $+\infty$ which means the received always decides x_1 was sent.)

(c)

$$
\begin{aligned}
\Pr\{\text{error}|X=x_1\} &= \Pr\Big\{Y \geq \frac{x_1+x_2}{2}\Big| X = x_1\Big\} \\
&= Q\Big(\frac{x_2-x_1}{2\sigma}\Big).
\end{aligned}
$$

since conditioned on $\{X = x_1\}$, Y is a $\mathcal{N}(x_1, \sigma^2)$ random variable. We can similarly show that

$$\Pr\{\text{error}|X=x_2\} = Q\Big(\frac{x_2-x_1}{2\sigma}\Big).$$

Therefore, the probability of error of the receiver equals

$$Q\Big(\frac{x_2-x_1}{2\sigma}\Big).$$

(d) Since we have n independent uses of the channel, conditioned on $\{\mathbf{X} = \mathbf{x}\}$ the output vector $\mathbf{Y}$ has distribution

$$f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{n} f_{Y|X}(y_i|x_i)$$

Thus, in particular

$$
\begin{aligned}
f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}_1) &= \prod_{i=1}^{n} f_{y|x}(y_i|x_1) \\
&= \frac{1}{\sigma^2(2\pi)^{n/2}} \exp\Big\{-\sum_{i=1}^{n} \frac{(y_i-x_1)^2}{2\sigma^2}\Big\} \\
&= \frac{1}{\sigma^2(2\pi)^{n/2}} \exp\Big\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2\Big\} \exp\Big\{\frac{1}{\sigma^2} x_1 \sum_{i=1}^{n} y_i - \frac{n x_1^2}{2\sigma^2}\Big\}
\end{aligned}
$$

Likewise,

$$f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}_2) = \frac{1}{\sigma^2(2\pi)^{n/2}} \exp\Big\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2\Big\} \exp\Big\{\frac{1}{\sigma^2} x_2 \sum_{i=1}^{n} y_i - \frac{n x_2^2}{2\sigma^2}\Big\}.$$

Therefore, by Fisher–Neyman factorization,

$$T(\mathbf{y}) = \sum_{i=1}^{n} y_i$$

is a sufficient statistic for the hypothesis testing problem.

(e) The optimal decision rule will be

$$\hat{\mathbf{X}}_{\mathrm{MAP}}(\mathbf{y}) = \begin{cases} \mathbf{x}_1 & \text{if } T(\mathbf{y}) \le n\Big(\dfrac{x_1 + x_2}{2}\Big) \\ \mathbf{x}_2 & \text{if } T(\mathbf{y}) \ge n\Big(\dfrac{x_1 + x_2}{2}\Big). \end{cases}$$

Given $\{\mathbf{X} = \mathbf{x}_1\}$, $T(\mathbf{Y}) = \sum_{i=1}^{n} Y_i$ has a $\mathcal{N}(nx_1, n\sigma^2)$ distribution, therefore

$$\Pr\{\text{error}|\mathbf{X} = \mathbf{x}_1\} = \Pr\Big\{T(\mathbf{y}) \ge n\frac{x_1 + x_2}{2}\Big\} = Q\Big(\sqrt{n} \cdot \frac{x_2 - x_1}{2\sigma}\Big)$$

Similarly,

$$\Pr\{\text{error}|\mathbf{X} = \mathbf{x}_2\} = Q\Big(\sqrt{n} \cdot \frac{x_2 - x_1}{2\sigma}\Big)$$

Therefore, the probability of error of the receiver equals

$$Q\Big(\sqrt{n} \cdot \frac{x_2 - x_1}{2\sigma}\Big)$$

Since $Q(\alpha) \le \exp(-\alpha^2/2)$ the above decays exponentially fast in n.

Solution 4.

(a) This receiver is supposed to form the observable by computing the correlation between the received signal and each of the sent signals. In order for $(U_1, \dots, U_4)$ to be a sufficient statistic we must have

$$U_i = \langle Y, x_i \rangle = \int_{-\infty}^{+\infty} Y(\tau) x_i(\tau)\, d\tau = \int_0^4 Y(\tau) x_i(\tau)\, d\tau.$$

From the theory of linear systems we know that U_is in the receiver will be equal to

$$U_i = \int Y(\tau) x_i\big(4 - (t_0 - \tau)\big)\, d\tau = \int Y(\tau) x_i(\tau - t_0 + 4)\, d\tau,$$

thus, by setting $t_0 = 4$ we get the desired sufficient statistics.

(b) Since the hypotheses are equally likely and x_1, x_2, x_3, and x_4 all have equal energy the optimal decision rule is simply

$$\hat{\imath}_{\mathrm{MAP}}(U_1, \dots, U_4) = \arg\max_i U_i$$

(That is, the receiver declares $x_i(t)$ is sent is if U_i is the largest among $U_1, \cdots, U_4$.)

(c) In this basis the signal representations are $\mathbf{x}_1 = (2, 0, 0, 2)$, $\mathbf{x}_2 = (0, 2, 2, 0)$, $\mathbf{x}_3 = (2, 0, 2, 0)$, and $\mathbf{x}_4 = (0, 2, 0, 2)$.

(d) The union bound is expressed in terms of the pairwise distances $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ between the signals since

$$\Pr\{\text{error}|\mathbf{x}_i\} \leq \sum_{j \neq i} Q\left(\frac{d_{ij}}{2\sigma}\right).$$

From (c) we observe that $d_{12}^2 = d_{34}^2 = 16$ and $d_{13}^2 = d_{14}^2 = d_{23}^2 = d_{24}^2 = 8$, hence

$$\Pr\{\text{error}|\mathbf{x}_i\} \leq 2Q\left(\frac{2}{\sqrt{N_0}}\right) + Q\left(\frac{2\sqrt{2}}{\sqrt{N_0}}\right)$$

Since the above does not depend on i, it also bounds the average error probability.

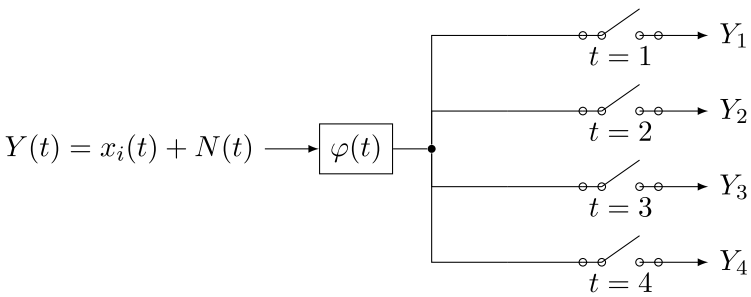
(e) Instead of correlating the received signal with each of the waveforms in the signal set, the receiver can project it onto the space spanned by an orthonormal basis for the signal set. Namely, to compute

$$Y_i = \langle Y, \varphi_i \rangle = \int Y(t)\varphi_i(t)\, dt, i = 1, 2, 3, 4.$$

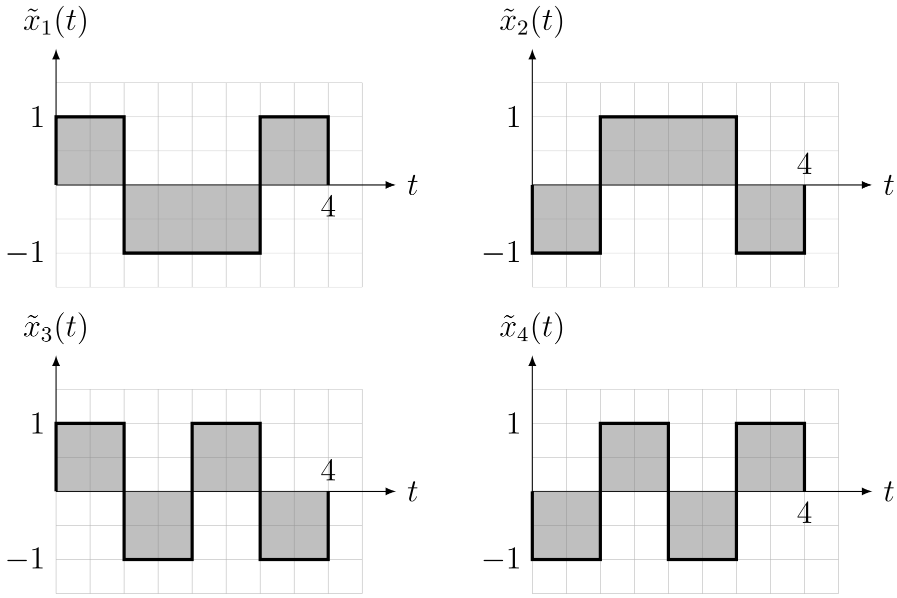
In this case the optimal decision in terms of $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$ will be the minimum distance decision,

$$\hat{\mathbf{x}}_{\text{MAP}}(\mathbf{y}) = \underset{\mathbf{x} \in \{\mathbf{x}_1, \ldots, \mathbf{x}_4\}}{\arg\min} \|\mathbf{y} - \mathbf{x}\|.$$

Moreover, since the basis signal are shifted versions of $\varphi(t)$ all the projections $\langle Y, \varphi_i \rangle$ can be computed using a single filter with impulse response $\varphi(1-t) = \varphi(t)$ by sampling its output at different times:



(f) The average of four signals is $\bar{x}(t) = \frac{1}{4}\sum_{i=1}^{4} x_i(t) = \mathbb{1}\{0 \leq t \leq 4\}$. We can obtain a minimum energy signal set by setting $\tilde{x}_i(t) = x_i(t) - \bar{x}(t)$.



Note that in the new signal set $\tilde{x}_2(t) = -\tilde{x}_1(t)$ and $\tilde{x}_4(t) = -\tilde{x}_3(t)$. Furthermore the signals $\tilde{x}_1(t)$ and $\tilde{x}_3(t)$ are orthogonal. Thus, using the orthonormal basis $\tilde{\varphi}_1(t) = \frac{\tilde{x}_1(t)}{\|\tilde{x}_1\|} = \frac{1}{2}\tilde{x}_1(t)$ and $\tilde{\varphi}_2(t) = \frac{\tilde{x}_3(t)}{\|\tilde{x}_3\|} = \frac{1}{2}\tilde{x}_3(t)$, the signal representations will be $\tilde{\mathbf{x}}_1 = (2, 0)$, $\tilde{\mathbf{x}}_2 = (-2, 0)$, $\tilde{\mathbf{x}}_3 = (0, 2)$, $\tilde{\mathbf{x}}_4 = (0, -2)$. These codewords correspond to those of a 4-PSK constellation.

The error probability of an optimal receiver for this set is

$$1 - \left[1 - Q\left(\frac{2}{\sqrt{N_0}}\right)\right]^2 = 2Q\left(\frac{2}{\sqrt{N_0}}\right) - Q\left(\frac{2}{\sqrt{N_0}}\right)^2$$

Moreover, since translations of a signal set do not change the probability of error, the error probability of the receiver in (b) is equal the same as above.