ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 22 | Advanced Digital Communications
Homework 9 | Nov. 21, 2016

PROBLEM 1. Given L independently faded observations of one and the same transmitted signal, select only the best channel and disregard all other observations. This is clearly a suboptimal approach, but has a low complexity and still achieves a diversity order L.

(a) What is the effective SNR Γ in this case?

(b) Compute the cumulative distribution function (cdf) $\Pr\{\Gamma \leq \gamma\}$ and use it to prove that the pdf of Γ is given by

$$f_\Gamma(\gamma) = \begin{cases} \frac{L}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}}(1 - e^{-\gamma/\overline{\gamma}})^{L-1} & \text{if } \gamma \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\gamma} = \frac{\mathcal{E}_c}{N_0/2}$ is the average effective SNR.

(c) Show that the average bit error probability is

$$\overline{P_\mathrm{e}} = \frac{1}{2} \sum_{l=0}^{L} (-1)^l \binom{L}{l} \sqrt{\frac{\overline{\gamma}}{2l + \overline{\gamma}}}.$$

(d) Prove by induction that for every non-negative integer n,

$$\lim_{x\to\infty} x \left(\left(\frac{x}{x-1} \right)^n - 1 \right) = n.$$

(e) Use (d) to approximate $\sqrt{\frac{\overline{\gamma}}{2l+\overline{\gamma}}}$ and show that the diversity order is indeed L.

PROBLEM 2. In your lecture notes, Section 5.5, we consider a system with two transmit and one receive antenna:

$$Y[n] \quad = \quad H_1 x_1[n] + H_2 x_2[n] + Z[n]$$

with the Alamouti scheme. Now we wish to extend this scheme to a system with two transmit and R receive antennas:

$$\begin{pmatrix} Y_1[n] \\ Y_2[n] \\ \vdots \\ Y_R[n] \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \\ \vdots & \vdots \\ H_{R1} & H_{R2} \end{pmatrix} \begin{pmatrix} x_1[n] \\ x_2[n] \end{pmatrix} + \begin{pmatrix} Z_1[n] \\ Z_2[n] \\ \vdots \\ Z_R[n] \end{pmatrix}$$

where we assume that $H_{i,1}$ and $H_{j,2}$ are independent circularly symmetric complex Gaussians with unit variance for any $i, j = 1, \dots, R$.

(a) Since there are $2R$ independent channel realizations, $H_{i,1}$ and $H_{j,2}, i, j = 1, \dots, R$, it might be tempting to believe that in a single channel use (meaning that we only transmit $x_1[n], x_2[n]$ for one time instance n), we can obtain a diversity of $2R$. Explain why this is not possible.

(b) Now, suppose we allow to code over *two* consecutive channel uses. Argue that we can now easily obtain a diversity of $2R$ if we transmit only a single information symbol, but we transmit it twice. Communications engineers like to refer to this as using only one of two possible "degrees of freedom." Be very specific, i.e., indicate how to select the two transmitted signals.

(c) Thus, we can obtain a diversity of $2R$ by exploiting only half of the possible degrees of freedom. Or, we can stick to diversity R and use all degrees of freedom. Typically, we expect a trade-off between using signal dimensions to enable degrees of freedom and using signal dimensions to obtain diversity. However, for the considered system, we can show that the Alamouti scheme lets you have both of them. The transmitting scheme is the same as for the system with one receive antenna in the lecture notes. To remind you, denoting the two (complex-valued) information symbols by u_1 and u_2, we transmit

$$\begin{aligned} x_1[0] = u_1 \quad &\text{and} \quad x_2[0] = u_2, \\ x_1[1] = -u_2^* \quad &\text{and} \quad x_2[1] = u_1^*. \end{aligned}$$

Show that with this trick, and the assumption that the channel remains the same for the two channel uses, *both* information symbols experience diversity $2R$. Thus, under the Alamouti scheme, for each symbol, we attain one degrees of freedom *and* diversity $2R$.

Problem 3. Consider a 2×2 MIMO system with inter-symbol interference (ISI), characterized by

$$\begin{aligned} y_1[n] &= x_1[n] + \frac{1}{2}x_1[n-1] + \frac{1}{2}x_1[n-3] - \frac{1}{2}x_2[n-1] - \frac{1}{2}x_2[n-3] + w_1[n] \\ y_2[n] &= \frac{1}{3}x_1[n-1] + \frac{1}{6}x_1[n-2] + x_2[n] - \frac{1}{2}x_2[n-2] + w_2[n], \end{aligned}$$

where $x_1[n]$ is the signal transmitted from the first antenna and $x_2[n]$ is the signal transmitted from the second antenna. As usual, $w_1[n]$ and $w_2[n]$ are independent circularly symmetric AWGN of variance N_0.

We would like to use OFDM to convert the original channel into a set of equivalent channels *without* ISI, but of different qualities.

(a) Describe your overall FFT-OFDM system design for general OFDM length N. Use both *pictures* and *mathematical formulas.* Provide formulas for the gains of the equivalent channels.

(b) For the special case $N = 4$ (see lecture notes on FFT-OFDM), give explicit numerical values. You may want to use numerical tools to compute the values.

(c) For $N = 4$ and with 2 transmit antennas, we thus have a total of $4 \times 2 = 8$ channel inputs (and the same number of channel outputs). Write the full 8×8 system, explicitly specifying the matrix. If we know the channel at the transmitter, then, as shown in the lecture notes, Section 5.7.1, we would like to diagonalize the channel. Find the 8 singular values and explicitly state the corresponding model with 8 parallel channels.

Problem 4. Consider a 2×2 MIMO system with inter-symbol interference (ISI), characterized by

$$\begin{aligned} y_1[n] &= h_{10}x_1[n] + h_{11}x_1[n-1] + g_{20}x_2[n] + g_{21}x_2[n-1] + w_1[n] \\ y_2[n] &= g_{10}x_1[n] + g_{11}x_1[n-1] + h_{20}x_2[n] + h_{21}x_2[n-1] + w_2[n], \end{aligned}$$

where h_{ij} and g_{ij} are independent Rayleigh fading (that is, circularly symmetric complex-valued Gaussians of variance 1).

As in the previous problem, we use FFT-OFDM. As you can verify, a cyclic prefix of length 1 is enough to get rid of the ISI. In this problem, we only study the simplest possible case, namely, when $N = 2$.

(a) Much like in the previous problem, we again observe that the OFDM channel 0 will simply be a 2×2 MIMO channel of the form

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} H_{11} & G_{12} \\ G_{21} & H_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

(and an analogous insight applies to OFDM channel 1, but let us only concentrate on OFDM channel 0 for now). Argue that H_{11}, H_{22}, G_{12}, and G_{21} are independent circularly symmetric complex-valued Gaussian random variables of variance 2. Note that this requires only a small number of formulas, and no lengthy calculations.

(b) Continuing (a), suppose now that we use a *zero-forcing filter* at the receiver. We transmit a single bit by setting X_1 as $\pm\sqrt{\mathcal{E}}$, as usual; we recover the bit simply by considering only the first entry of the zero-forced output (denoted as $\tilde{Y}_1$ in the Lecture Notes) and thresholding this at zero. Develop a formula for the average error probability of recovering this bit, averaged over H_{11}, H_{22}, G_{12}, and G_{21}. Write your formula as an integral (which you could then numerically approximate with numerical tools).

PROBLEM 5. In this problem, we study the case where there is no Channel State Information (CSI) at the receiver, that is, the value of the fading H is unknown. To do this, we consider Orthogonal Signaling exactly as in Equations (5.31)–(5.33) of your lecture notes. However, since the receiver does *not* know the value of H, it cannot proceed as in Equations (5.34) and (5.35).

(a) Express the pdfs of $(Y[0], Y[1])$ when the transmitted signal is $(x[0], x[1]) = (\sqrt{\mathcal{E}_c}, 0)$ and when that the transmitted signal is $(x[0], x[1]) = (0, \sqrt{\mathcal{E}_c})$.

(b) Using the likelihood ratio, derive the ML detector. Simplify it as much as possible.

(c) Determine the resulting average error probability.

(d) What is the diversity order in this setting?

For the rest of the problem, we want to extend to the case of diversity. Specifically, suppose now that we have L independently faded copies. That is, the transmitted signal is exactly as in Equation (5.33) in the lecture notes, but we have L received signal pairs,

$$Y_\ell[0] = H_\ell x[0] + Z_\ell[0] \quad \text{and} \quad Y_\ell[1] = H_\ell x[1] + Z_\ell[1],$$

for $\ell = 1, 2, \ldots, L$, where H_ℓ are independent Rayleigh fading. As before, the receiver does not know the realizations of the fading.

(e) Argue that we can definitely obtain a diversity order of $L/2$.

The goal of the rest of the problem is to derive the diversity of the ML detector. In class, we already encountered the PDF of the sum of the squares of $2L$ i.i.d. Gaussians with mean zero and variance 1 (which is also the sum of L i.i.d. exponential random variables with mean 2), which is the χ-squared distribution with $2L$ degrees of freedom, given by

$$\frac{1}{2^L L!} x^{L-1} e^{-x/2}.$$

(f) Prove that the probability of error of the ML detector is of the form

$$P_{\mathrm{e}} = P(W \leq \alpha),$$

where W is the ratio of two independent χ-squared random variables. Find the parameter α.

(g) The ratio of two independent χ-squared random variables with $2L$ degrees of freedom has the so-called F-distribution, with the following CDF:

$$F_{2L,2L}(x) = I_{\frac{x}{x+1}}(L, L),$$

where $I_{\frac{x}{x+1}}(L, L)$ is the regularized incomplete beta function. Despite its intimidating name, the regularized incomplete beta function satisfies the following simple recursive equation:

$$I_x(a, b+1) = I_x(a, b) + \frac{x^a(1-x)^b}{bB(a, b)} \tag{1}$$

where $B(a, b)$ is a strictly positive function. It also satisfies

$$I_x(a, 1) = x^a. \tag{2}$$

Write the probability of error that you found in (f) in terms of the regularized incomplete beta function. Apply the recursive equation (1) $L - 1$ times and finally use equation (2) in order to find a closed form expression for the probability of error.

(h) By upper-bounding each term of the expression that you found in (g), argue that

$$P_{\mathrm{e}} \leq \frac{\theta}{\overline{\gamma}^K}$$

where $\overline{\gamma} = \frac{\mathcal{E}_c}{N_0/2}$ and θ is a constant that depends *only* on L. What is the value of K? What is the diversity of the ML detector?