ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 20 Advanced Digital Communications
Homework 8 Nov. 14, 2016

PROBLEM 1. We have seen in class that the average bit error probability of antipodal signaling over Rayleigh flat-fading is

$$\overline{P_{\mathrm{e}}} = \int_0^\infty Q(\sqrt{\gamma}) \frac{1}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}} \, \mathrm{d}\gamma,$$

where $\overline{\gamma} = \frac{\mathcal{E}_c}{N_0/2}$. Show that this integral has the closed form solution

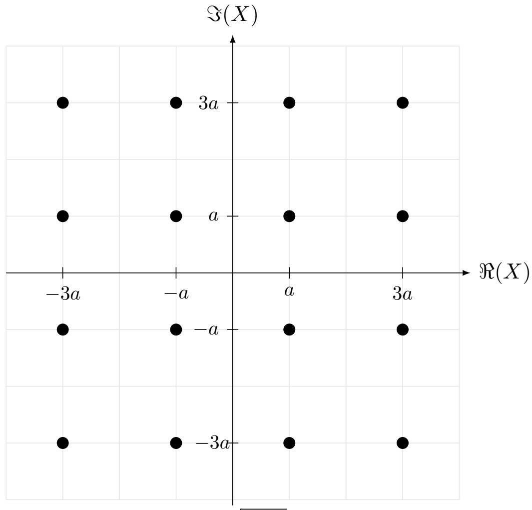
$$\overline{P_{\mathrm{e}}} = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right).$$

PROBLEM 2. Consider the complex-valued channel

$$Y = hX + Z$$

where Z is circularly symmetric complex Gaussian noise with variance N_0. We assume that X is chosen uniformly from the following 16-QAM constellation with average energy $\mathcal{E}$. It can be shown by a direct calculation that the ML detection for X with a fixed channel h has the following error probability

$$P_{\mathrm{e}} = 3Q\left(\sqrt{\frac{1}{10}\frac{|h|^2\mathcal{E}}{N_0/2}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{1}{10}\frac{|h|^2\mathcal{E}}{N_0/2}}\right).$$



$a = \sqrt{\mathcal{E}/10}$

Now consider a fading channel

$$Y = HX + Z$$

where H is circularly symmetric complex Gaussian with unit variance. The channel input X is chosen uniformly from the 16-QAM constellation described above.

(a) Define $\overline{\gamma} := \frac{\mathcal{E}}{N_0/2}$ and compute the average error probability $\overline{P_{\mathrm{e}}}$ in terms of $\overline{\gamma}$ averaging over all fading states H.
Hint. You can upper bound the error probability of the ML decoder, given above, by neglecting the term with the squared Q-function to simplify the calculations.

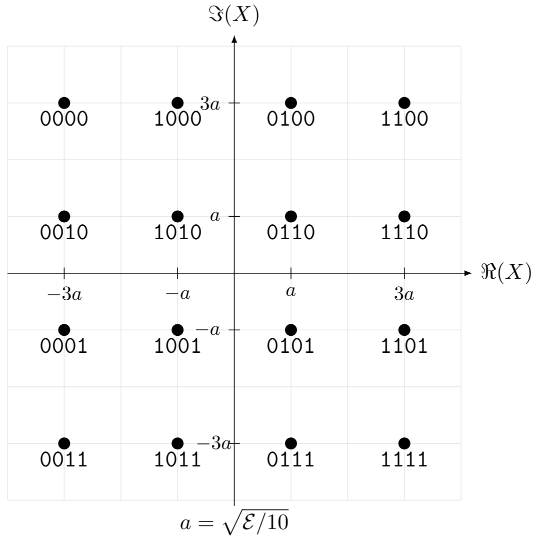
(b) Define the *diversity order* to be

$$L := -\lim_{\overline{\gamma}\to\infty} \frac{\log \overline{P_{\mathrm{e}}}}{\log \overline{\gamma}},$$

as in the lecture notes. What is the diversity order in this case?

(c) Compare the result in (a) to the BPSK case we studied in the class where we have shown that the average error probability $\overline{P_{\mathrm{e}}} \approx \frac{1}{2\overline{\gamma}}$ for high SNR. How large is the difference? Explain why it is the case.

(d) So far we have only studied the *symbol error probability*. Now we want to compute the*bit error probability*. This depends on how you map the 4 bits to the 16 symbols of the 16-QAM constellation. Suppose the bits are mapped as shown in the following figure. In order to simplify the analysis, we will only consider the error events that a symbol is confused with its nearest neighbors. For example, when we consider the symbol with bit assignment `1010` in the detection, we only care about the error events that this symbol is confused with symbols with minimum distance $2\sqrt{\mathcal{E}/10}$ to it, i.e., the symbols with bits assignment `1000`, `0010`, `1001`, or `0110`.



Under this assumption, calculate the bit error probability for i-th bit, $i = 1, \dots, 4$ when using the above bit assignment and express the average bit error probability averaging over the four bits for a fixed channel h in terms of $\bar{\gamma}$. Finally, for the fading channel, compute the average bit error probability by averaging again over the channel and the diversity.

(e) Can you come up with another bits assignment scheme which, under the same assumption of error events, has a better average error probability than the naive scheme?

PROBLEM 3. Consider the Rayleigh fading channel

$$Y[n] = H[n]X[n] + Z[n]$$

for $n = 1, \dots, L$ where L is an odd number. We assume $H[n]$ are independent circularly-symmetric complex Gaussian random variables of unit variance. The noise $Z[n]$ is circularly-symmetric complex additive white Gaussian noise whose real and imaginary parts each have variance $N_0/2$.

We want to convey one single bit of information $b \in \{0, 1\}$ with L uses of this channel. If the information bit is $b = 0$, we transmit $\boldsymbol{X} = [-2\sqrt{\mathcal{E}}, \dots, -2\sqrt{\mathcal{E}}]$ and if $b = 1$, we transmit $\boldsymbol{X} = [2\sqrt{\mathcal{E}}, \dots, 2\sqrt{\mathcal{E}}]$. Consider doing a *hard decision* on each component of the channel output separately, and then fuse these hard decisions for the estimation. Derive the error probability and express it in terms of $\mathcal{E}$ and N_0. What is the resulting diversity order?