ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 16 Advanced Digital Communications

Homework 7 (Midterm Exam of 2015) Nov. 7, 2016

PROBLEM 1. Consider the usual discrete-time ISI channel model considered in class, with

$$D(z) = \frac{1}{(1-\alpha z)(1-\alpha z^{-1})},$$

where α is a real-valued constant satisfying $|\alpha| < 1$, meaning that the channel is given by

$$U[n] = \sum_{k=-\infty}^{\infty} d[k]I[n-k] + V[n],$$

where $V[n]$ is additive Gaussian noise with power spectral density $S_V(z) = \frac{N_0}{2}D(z)$.

(a) If we apply the zero-forcing equalizer $1/D(z)$ to $U[n]$, then we have seen in class that the resulting overall channel becomes $\hat{I}_{\mathrm{ZF}}[n] = I[n] + \tilde{V}[n]$. Find the variance of the noise $\tilde{V}[n]$, that is, find $\mathbb{E}[\tilde{V}[n]^2]$ (as a function of N_0 and α).

(b) Assuming that we use BPSK (that is, the information symbols $I[n]$ are independent of each other, and are $\pm\sqrt{\mathcal{E}}$ with uniform priors) and we separately slice each received symbol $\hat{I}_{\mathrm{ZF}}[n]$ at zero (exactly as in class), give the formula for the resulting bit error probability in terms of the problem parameters, using the Q-function.

(c) Next, we would like to understand how suboptimal the Zero-forcing solution is. To this end, as in class, let us apply the whitening filter $(1 + \alpha z)$ to the channel output $U[n]$. Working out the details, we find that the resulting equivalent channel is given by

$$S[n] = \sum_{k=0}^{\infty} \alpha^k I[n-k] + W[n],$$

where $W[n]$ is the usual (real-valued) AWGN of variance $N_0/2$.

To find the minimum possible bit error probability is difficult for this model. Instead, we will now derive a *lower bound* on the bit error probability — that is, an expression that is *lower* than what we could ever hope to achieve. In order to obtain such a lower bound, we idealize the situation: let us suppose that we set $I[n] = 0$ for all $n \neq 0$, and in fact only send a single bit by setting $I[0] = \sqrt{\mathcal{E}}$ if the bit is one, and $I[0] = -\sqrt{\mathcal{E}}$ if the bit is zero. Using the full received sequence $S[n]$, for $-\infty < n < \infty$, find a formula for the error probability of the ML detector for recovering the single transmitted bit.

(d) Compare the error probability formulas from parts (b) and (c). Specifically, in order to attain a certain desired target error probability (call it P_0), how much more transmit energy does the Zero-forcing solution from part (b) need than the lower bound from part (c)? Discuss this as a function of the parameter α.

PROBLEM 2. Consider a single transmitter and two receivers, which we will refer to as users A and B, respectively. The transmitted signal is $x[n]$, and the received signals are

$$y_A[n] = x[n] + \frac{1}{2}x[n-1] + \frac{1}{2}x[n-3] + w_A[n], \qquad y_B[n] = \sqrt{\frac{3}{2}}x[n] + \sqrt{\frac{3}{2}}x[n-2] + w_B[n],$$

where $w_A[n]$ and $w_B[n]$ are independent of each other and the usual circularly symmetric complex-valued Gaussian noises of variance N_0. The transmitter uses FFT-OFDM.

(a) Determine the *minimum* length of cyclic prefix needed so that both users have an ISI-free equivalent channel.

(b) For $N = 4$, determine the equivalent four channel gains from the transmitter to user B (that is, the coefficients that were denoted as H_0, H_1, H_2, and H_3 in the class).

(c) Suppose that the transmitter is only sending a single bit: If the bit is one, then the transmitter sends $\sqrt{\mathcal{E}}$ through all 4 channels (that is, $X_0 = X_1 = X_2 = X_3 = \sqrt{\mathcal{E}}$), and if the bit is zero, then the transmitter sends $-\sqrt{\mathcal{E}}$ through all 4 channels.

For user A, determine the ML detector of the single transmitted bit, given all four channel outputs and give a formula for the resulting error probability.

(d) Suppose that the transmitter has one bit for user A and one bit for user B. In each of the 4 channels, we will send only either $\sqrt{\mathcal{E}}$ or $-\sqrt{\mathcal{E}}$, exactly like in part (c). But this time, we choose a subset of the channels to send one bit to user A, and a different subset of the channels to send the other bit to user B. Note that user A is only interested in her bit, and does not attempt to recover the bit for user B, and vice versa. The goal is to ensure that both users experience the same (or almost the same) error probability in recovering their respective bit of interest, and that this error probability is as small as possible. How should the channels be divided between the users? And what are the resulting error probabilities for users A and B, respectively?

(e) Exactly like in part (d), suppose that the transmitter has one bit for user A and one bit for user B. But by contrast, we now suppose that we have a total energy of $4\mathcal{E}$ that we are allowed to split any way we want between the two users. More precisely, in each channel, we still do BPSK, but we select the energy for the BPSK in a clever way. Then, each channel is assigned either to user A or to user B (or it can be left unused), and for each channel, we have to determine what fraction of the total energy to use. As before, the goal is to *simultaneously* have the smallest possible error probability for both users in recovering their respective bits. Also give the formulas for the attained error probabilities.

PROBLEM 3.

(a) Consider the scalar AWGN channel $Y = x + Z$, where x is uniformly selected from $\{\pm\sqrt{\mathcal{E}}\}$ and Z is the usual AWGN with power $N_0/2$. Suppose that we quantize Y as:

$$D = \begin{cases} 1, & \text{if } Y \geq \theta\sqrt{\mathcal{E}}, \\ *, & \text{if } -\theta\sqrt{\mathcal{E}} \leq Y < \theta\sqrt{\mathcal{E}}, \\ 0, & \text{if } Y < -\theta\sqrt{\mathcal{E}}, \end{cases}$$

where θ is an arbitrary (non-negative) constant. For future convenience, let us define:

$$\alpha := \Pr\{D = 0 | x = \sqrt{\mathcal{E}}\} \qquad \beta := \Pr\{D = * | x = \sqrt{\mathcal{E}}\}.$$

Calculate the values of α and β as a function of the problem parameters, using the Q-function. Finally, argue that $\Pr\{D = * | x = -\sqrt{\mathcal{E}}\} = \beta$ and $\Pr\{D = 1 | x = -\sqrt{\mathcal{E}}\} = \alpha$.

(b) Now consider the vector AWGN channel of length N, i.e., $\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{Z}$, where $\boldsymbol{Z}$ is the usual AWGN of variance $N_0/2$. Suppose that $\boldsymbol{x} \in \{\pm\sqrt{\mathcal{E}}(1, 1, \ldots, 1)\}$ (with uniform priors). Next, we apply the hard-decision detector from part (a) separately to each entry in the vector $\boldsymbol{Y}$ to obtain the vector $\boldsymbol{D}$, whose entries are thus either 0, 1, or $*$. Derive the ML detector based on $\boldsymbol{D}$. Simplify it as much as possible.